

Derivation of the exact NSVZ β -function in $N = 1$ SQED, regularized by higher derivatives, by direct summation of Feynman diagrams

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Abstract

For $N = 1$ supersymmetric quantum electrodynamics, regularized by higher derivatives, a method for summation of all Feynman diagrams defining the β -function is presented. Using this method we prove that the β -function is given by an integral of a total derivative, which can be easily calculated. It is shown that surviving terms give the exact NSVZ β -function. The results are compared with the explicit three-loop calculation.

1 Introduction.

Quantum corrections in supersymmetric theories (and, in particular, the β -function) were studied for a long time. The exact β -function for $N = 1$ supersymmetric theories,

$$\beta(\alpha) = -\frac{\alpha^2 \left[3C_2 - T(R) + C(R)_i^j \gamma_j^i(\alpha)/r \right]}{2\pi(1 - C_2\alpha/2\pi)}, \quad (1)$$

was found in [1]. Certainly, a β -function in supersymmetric theories was also calculated explicitly in the lowest loops. Most calculations were made with the dimensional reduction [2] in \overline{MS} -scheme [3]. The NSVZ β -function agrees with these calculations in the one- and two-loop approximations. In order to obtain NSVZ β -function in higher loops it is necessary to perform a special redefinition of the coupling constant [4]. The possibility of such a redefinition is very nontrivial [5]. However, it is well known [6] that the dimensional reduction is not self-consistent. (The dimensional regularization [7] breaks the supersymmetry and is not convenient for calculations in supersymmetric theories.) Ways allowing to avoid such problems are discussed in the literature [8].

Other regularizations are also sometimes applied for calculations in supersymmetric theories. For example, in Ref. [9] a two-loop β -function of the $N = 1$ supersymmetric Yang–Mills theory was calculated with the differential renormalization [10]. Some

calculations were made with the higher covariant derivative regularization, proposed in [11], which was generalized to the supersymmetric case in Ref. [12] (another variant was proposed in Ref. [13]). Usually integrals arising with the higher covariant derivative regularization can not be calculated analytically in higher loops. That is why this regularization was applied for explicit calculations rather rarely. In particular, the first calculation of quantum corrections for the (non-supersymmetric) Yang–Mills theory was made in Ref. [14]. Taking into account corrections, made in subsequent papers [15], the result for the β -function appeared to be the same as the well-known result obtained with the dimensional regularization [16]. In principle, it is possible to prove that in the one-loop approximation calculations with the higher covariant derivative regularization always agree with the results of calculations with the dimensional regularization [17]. Some calculations in the one- and two-loop approximations were made for various theories [18, 19] with a variant of the higher covariant derivative regularization, proposed in [20]. The structure of the corresponding integrals was discussed in Ref. [19].

The three-loop β -function for the $N = 1$ SQED, regularized by higher derivatives, was calculated in [21]. This calculation shows that the integrals defining the β -function are integrals of total derivatives. A two-loop calculation made with the dimensional reduction in the $N = 1$ SQED and revealing a similar feature was presented in Ref. [22], where the factorization of integrands into total derivatives is explained in all loops using a special technique, based on the covariant Feynman rules in the background field method [23]. This factorization allows to calculate one of the loop integrals analytically. As a consequence, (in the $N = 1$ SQED) integrals defining the β -function are reduced to integrals defining the anomalous dimension, producing NSVZ β -function. In particular, with the higher derivative regularization it is not necessary to perform a redefinition of the coupling constant. The factorization of integrands into total derivatives seems to be a general feature of supersymmetric theories. In the two-loop approximation it was verified for a general renormalizable $N = 1$ supersymmetric theory, regularized by higher covariant derivatives, in [24].

Using a usual supergraph technique [25, 26] an attempt to prove the factorization of integrands into total derivatives was made in Ref. [27], where a solution of the Ward identity was substituted into the Schwinger–Dyson equations. This allows to present a β -function as a sum of two contributions: The first one (calculated in Ref. [27] exactly to all orders) is given by an integral of a total derivative and is expressed in terms of the two-point function of the matter superfields. This contribution gives the exact NSVZ β -function. The second contribution is essential starting from the three-loop approximation and can be expressed in terms of a transversal part of a certain Green function. Explicit three-loop calculations [21] show that the second contribution to the β -function is also given by an integral of a total derivative and is equal to 0. It was conjectured [27] that this takes place in all loops. A partial four-loop verification of this statement were made in Ref. [28]. The factorization of the additional contribution into an integral of a total derivative was qualitatively explained in [29], using a technique proposed in Ref. [30]. In this paper we formulate these arguments in a rigorous form and prove that all integrals in the $N = 1$ SQED defining the β -function are integrals of total derivatives. Taking these integrals we obtain the exact NSVZ β -function without a redefinition of the coupling constant. The results are compared with the explicit three-loop calculation, made in [21].

The paper is organized as follows:

In Sec. 2 we recall basic information about the $N = 1$ supersymmetric electrodynamics

and its regularization by higher derivatives. In Sec. 3 we perform the integration over the matter superfields in the generating functional and introduce some notation. Then it is possible to construct formal expressions for a β -function and an anomalous dimension, encoding sums of Feynman diagrams. This is made in Sec. 4. In Sec. 5 we describe some tricks allowing to simplify the calculation of the Feynman diagrams. In the massless case these diagrams are calculated (exactly to all orders) in Sec. 6. In this section the sum of diagrams is reduced to integrals of total derivatives. From these integrals of total derivatives the exact NSVZ β -function is derived exactly to all orders in Sec. 7. A similar investigation for the Pauli–Villars contributions is made in Sec. 8. The result is also given by integrals of total derivatives. The obtained expressions are verified by the explicit three-loop calculation in Sec. 9. The results are summarized in the Conclusion. Some technical details are presented in the Appendixes.

2 $N = 1$ supersymmetric electrodynamics and its regularization by higher derivatives

The massless $N = 1$ supersymmetric electrodynamics is described by the following action:¹

$$S = \frac{1}{4e^2} \text{Re} \int d^4x d^2\theta W_a C^{ab} W_b + \frac{1}{4} \int d^4x d^4\theta \left(\phi^* e^{2V} \phi + \tilde{\phi}^* e^{-2V} \tilde{\phi} \right). \quad (2)$$

Here ϕ and $\tilde{\phi}$ are chiral matter superfields, and V is a real scalar superfield, which contains the gauge field A_μ as a component. The superfield W_a is a supersymmetric analog of the gauge field strength. In the Abelian case it is defined by

$$W_a = \frac{1}{4} \bar{D}^2 D_a V. \quad (3)$$

(D_a and \bar{D}_a are the right and left supersymmetric covariant derivatives, respectively.)

In order to regularize the theory we add to the action a term with higher derivatives. Then the action can be written as

$$S_{\text{reg}} = \frac{1}{4e^2} \text{Re} \int d^4x d^2\theta W_a C^{ab} R\left(\frac{\partial^2}{\Lambda^2}\right) W_b + \frac{1}{4} \int d^4x d^4\theta \left(\phi^* e^{2V} \phi + \tilde{\phi}^* e^{-2V} \tilde{\phi} \right), \quad (4)$$

where the function R satisfies the following conditions:

$$R(0) = 1; \quad R(\infty) = \infty. \quad (5)$$

For example, it is possible to choose

$$R\left(\frac{\partial^2}{\Lambda^2}\right) = 1 + \frac{\partial^{2n}}{\Lambda^{2n}}. \quad (6)$$

¹In our notation $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$; $\theta^a \equiv \theta_b C^{ba}$; θ_a and $\bar{\theta}_a$ denote the right and left components of θ , respectively.

In the Abelian case the superfield W_a is gauge invariant, so that action (4) is also gauge invariant.

Quantization of model (4) can be made by the standard way. For this purpose it is convenient to use the supergraph technique, described, for example, in textbooks [25, 26], and to fix the gauge invariance by adding the following terms:

$$S_{\text{gf}} = -\frac{1}{64e^2} \int d^4x d^4\theta \left(V D^2 \bar{D}^2 R \left(\frac{\partial^2}{\Lambda^2} \right) V + V \bar{D}^2 D^2 R \left(\frac{\partial^2}{\Lambda^2} \right) V \right). \quad (7)$$

After adding such terms, a kinetic term for the superfield V will have the simplest form

$$S_{\text{reg}} + S_{\text{gf}} = \frac{1}{4e^2} \int d^4x d^4\theta V \partial^2 R \left(\frac{\partial^2}{\Lambda^2} \right) V. \quad (8)$$

In the Abelian case, considered here, diagrams with ghost loops are absent.

Adding the higher derivative term does not remove divergences in the one-loop diagrams. In order to regularize them, it is necessary to insert Pauli-Villars determinants into the generating functional [31]. Therefore, the generating functional can be written as

$$Z = \int DV D\phi D\tilde{\phi} \prod_I \left(\det PV(V, M_I) \right)^{c_I} \exp \left(i(S_{\text{reg}} + S_{\text{gf}} + S_{\text{Source}}) \right), \quad (9)$$

where

$$S_{\text{Source}} = \int d^4x d^4\theta V J + \left(\int d^4x d^2\theta (\phi j + \tilde{\phi} \tilde{j}) + \text{h.c.} \right). \quad (10)$$

(It is necessary to substitute e in S_{reg} and S_{gf} by the bare coupling constant e_0 .) The Pauli-Villars determinants are given by

$$\left(\det PV(V, M) \right)^{-1} = \int D\Phi D\tilde{\Phi} e^{iS_{\text{PV}}}, \quad (11)$$

where²

$$S_{\text{PV}} \equiv \frac{1}{4} \int d^4x d^4\theta \left(\Phi^* e^{2V} \Phi + \tilde{\Phi}^* e^{-2V} \tilde{\Phi} \right) + \left(\frac{1}{2} \int d^4x d^2\theta M \tilde{\Phi} \Phi + \text{h.c.} \right). \quad (12)$$

The coefficients c_I satisfy conditions

$$\sum_I c_I = 1; \quad \sum_I c_I M_I^2 = 0. \quad (13)$$

Below we will assume, that $M_I = a_I \Lambda$, where a_I are constants. Insertion of the Pauli-Villars determinants allows to cancel remaining divergences in all one-loop diagrams.

The generating functional for connected Green functions and the effective action are defined by the standard way.

²Note that the Pauli-Villars action differs from the one used in [21] because here the ratio of the coefficients in the kinetic term and in the mass term does not contain the factor Z . Using terminology of Ref. [32], one can say that here we will calculate the canonical coupling α_c , while in Ref. [21] the holomorphic coupling α_h have been calculated. Certainly, after the renormalization the effective action does not depend on the definitions. However, the definitions used here are much more convenient.

3 Generating functional

In order to derive an exact β -function we perform explicit summation of the corresponding Feynman diagrams. However, for the rigorous proof it is desirable to obtain some formal expressions encoding sums of these diagrams. For this purpose we first perform the integration over the matter superfields in generating functional (9). This can be made, because the corresponding integral is Gaussian. It is easy to see that the result is

$$Z = \int DV \prod_I \left(\det PV(V, M_I) \right)^{c_I} \det(*) \det(\tilde{*}) \times \exp \left\{ i \int d^8x \left(\frac{1}{4e_0^2} V \partial^2 R(\partial^2/\Lambda^2) V - j \frac{D^2}{4\partial^2} * \frac{\bar{D}^2}{4\partial^2} j^* - \tilde{j} \frac{D^2}{4\partial^2} \tilde{*} \frac{\bar{D}^2}{4\partial^2} \tilde{j}^* \right) \right\}, \quad (14)$$

where

$$* \equiv \frac{1}{1 - (e^{2V} - 1) \bar{D}^2 D^2 / 16 \partial^2}, \quad \tilde{*} \equiv \frac{1}{1 - (e^{-2V} - 1) \bar{D}^2 D^2 / 16 \partial^2} \quad (15)$$

encode chains of propagators, connecting vertexes with the quantum gauge field.³ In order to obtain generating functional (14), we note that a solution of the motion equation for the chiral superfield ϕ

$$8j^* = D^2(e^{2V}\phi) = D^2\phi + D^2((e^{2V} - 1)\phi) \quad (16)$$

can be written as

$$\phi = - \left(1 - \frac{\bar{D}^2 D^2}{16 \partial^2} (e^{2V} - 1) \right)^{-1} \frac{\bar{D}^2}{2 \partial^2} j^* = \frac{\bar{D}^2 D^2}{16 \partial^2} * \frac{\bar{D}^2}{2 \partial^2} j^*. \quad (17)$$

Similar to derivation of Eq. (14) it is possible to perform integration over the Pauli–Villars fields. Introducing sources for the Pauli–Villars fields in expression (11), we can write the Pauli–Villars determinants in a similar form:

$$\det PV(V, M, \mathbf{j}_{\text{pv}}) = \left(\det(\star) \right)^{1/2} \exp \left\{ - \frac{i}{2} \int d^8x \mathbf{j}_{\text{pv}}^T A P \star A \mathbf{j}_{\text{pv}} \right\}, \quad (18)$$

where we use the following notation:

$$A \equiv \frac{1}{4\partial^2} \begin{pmatrix} D^2 & 0 & 0 & 0 \\ 0 & \bar{D}^2 & 0 & 0 \\ 0 & 0 & D^2 & 0 \\ 0 & 0 & 0 & \bar{D}^2 \end{pmatrix}; \quad \mathbf{j} = \begin{pmatrix} j \\ j^* \\ \tilde{j} \\ \tilde{j}^* \end{pmatrix}. \quad (19)$$

\star is defined by

$$\star \equiv \frac{1}{1 - I_0} \quad \text{with} \quad I_0 = \mathcal{V}P, \quad (20)$$

³This is a rigorous definition. In Ref. [29] $*$ was defined only qualitatively using Feynman rules. That is why some equations below differ from the corresponding equations in Ref. [29].

where

$$\mathcal{V} = \begin{pmatrix} 0 & (e^{2V} - 1) & 0 & 0 \\ (e^{2V} - 1) & 0 & 0 & 0 \\ 0 & 0 & 0 & (e^{-2V} - 1) \\ 0 & 0 & (e^{-2V} - 1) & 0 \end{pmatrix} \quad (21)$$

is a vertex contribution, and

$$P = \begin{pmatrix} 0 & \frac{\bar{D}^2 D^2}{16(\partial^2 + M^2)} & \frac{M \bar{D}^2}{4(\partial^2 + M^2)} & 0 \\ \frac{D^2 \bar{D}^2}{16(\partial^2 + M^2)} & 0 & 0 & \frac{M D^2}{4(\partial^2 + M^2)} \\ \frac{M \bar{D}^2}{4(\partial^2 + M^2)} & 0 & 0 & \frac{\bar{D}^2 D^2}{16(\partial^2 + M^2)} \\ 0 & \frac{M D^2}{4(\partial^2 + M^2)} & \frac{D^2 \bar{D}^2}{16(\partial^2 + M^2)} & 0 \end{pmatrix} \quad (22)$$

corresponds to propagators.

The contribution of ϕ and $\tilde{\phi}$ can be also written in a form similar to (18):

$$Z = \int DV \prod_I \left(\det PV(V, M_I) \right)^{c_I} \left(\det(\star) \right)^{1/2} \\ \times \exp \left\{ i \int d^8 x \frac{1}{4e_0^2} V \partial^2 R(\partial^2 / \Lambda^2) V + \frac{i}{2} \int d^8 x \mathbf{j}^T A P \star A \mathbf{j} \right\}, \quad (23)$$

where M inside \star should be set to 0.

It is also convenient to define

$$(I_1)_a \equiv [I_0, \theta_a]; \quad (\bar{I}_1)_a \equiv [I_0, \bar{\theta}_a]; \quad (I_2) \equiv \{[I_0, \theta_a], \theta^a\}; \\ (I_2)_{ab} \equiv \{[I_0, \bar{\theta}_a], \theta_b\}; \quad (\bar{I}_3)_b \equiv \{[I_0, \theta_a], \theta^a\}, \bar{\theta}_b]. \quad (24)$$

If, for example⁴, $I_0 = (e^{2V} - 1) \bar{D}^2 D^2 / 16 \partial^2$, then

$$(I_1)_a = (e^{2V} - 1) \frac{\bar{D}^2 D_a}{8 \partial^2}; \quad (\bar{I}_1)_a = (e^{2V} - 1) \frac{\bar{D}_a D^2}{8 \partial^2}; \quad (I_2) = (e^{2V} - 1) \frac{\bar{D}^2}{4 \partial^2}; \\ (I_2)_{ab} = (e^{2V} - 1) \frac{\bar{D}_a D_b}{4 \partial^2}; \quad (\bar{I}_3)_b = (e^{2V} - 1) \frac{\bar{D}_b}{2 \partial^2}. \quad (25)$$

Let us also describe some properties of \star , which will be useful below:

It is easy to see that

$$\star I_0 \star = -\star + \star^2, \quad (26)$$

⁴This expression can be used in the massless case, if only the field ϕ is present. In the general case I_0 is defined by Eq. (20).

and the following expansions take place:

$$\star = 1 + \sum_{n=1}^{\infty} (I_0)^n; \quad \ln \star = \sum_{n=1}^{\infty} \frac{1}{n} (I_0)^n; \quad \star^2 = \sum_{n=0}^{\infty} (n+1)(I_0)^n, \quad \text{etc.} \quad (27)$$

Therefore,

$$\begin{aligned} (\ln \star)_n &= \frac{1}{n} (\star)_n; & (\star^2)_n &= (n+1)(\star)_n; & (\star^3)_n &= \frac{(n+1)(n+2)}{2} (\star)_n; \\ (\star^4)_n &= \frac{(n+1)(n+2)(n+3)}{6} (\star)_n, \end{aligned} \quad (28)$$

where the subscript n denotes the n -th term.

4 Formal calculation of renormgroup function

Expression (14) allows to obtain a simple formal expression for an anomalous dimension. A Green function of the matter superfield is given by

$$\frac{\delta^2 \Gamma}{\delta \phi_x \delta \phi_y^*} = \frac{\bar{D}_x^2 D_x^2}{16} G(\partial_x^2) \delta_{xy}^8. \quad (29)$$

The corresponding inverse function, which by definition satisfies

$$\int d^8 y \frac{\delta^2 \Gamma}{\delta \phi_x \delta \phi_y^*} \frac{\bar{D}_y^2}{8 \partial^2} \left(\frac{\delta^2 \Gamma}{\delta \phi_z \delta \phi_y^*} \right)^{-1} = -\frac{1}{2} \bar{D}_x^2 \delta_{xz}^8, \quad (30)$$

is

$$\left(\frac{\delta^2 \Gamma}{\delta \phi_y^* \delta \phi_x} \right)^{-1} = -\frac{\bar{D}_x^2 D_x^2}{4 \partial^2} G^{-1}(\partial^2) \delta_{xy}^8 = -\frac{\delta^2 W}{\delta j_y^* \delta j_x} = \left\langle \frac{\bar{D}_x^2 D_x^2}{8 \partial^2} * \frac{\bar{D}_x^2 D_x^2}{8 \partial^2} \delta_{xy}^8 \right\rangle. \quad (31)$$

(In order to derive the last equality, we have differentiated generating functional (14) with respect to the sources.) The angular brackets denote the functional integration over the gauge superfield V . (The factors $\det(*)$ and $\det(\tilde{*})$ should be certainly included.) From this equation we obtain

$$\bar{D}_x^2 D_x^2 G^{-1}(\partial^2) \delta_{xy}^8 = \left\langle * \bar{D}_x^2 D_x^2 \delta_{xy}^8 \right\rangle. \quad (32)$$

This can be verified by applying to both sides the operator $-\bar{D}_x^2 D_x^2 / 16 \partial_y^2$. We will use expression (32) later.

It is also possible to construct an expression for a two-point Green function of the gauge superfield. For this purpose we use the equation

$$\Gamma_{\mathbf{V}}^{(2)} = \frac{1}{4e_0^2} \int d^8 x \mathbf{V} \partial^2 R \mathbf{V} + \frac{1}{2} \int d^8 x d^8 y \mathbf{V}_x \mathbf{V}_y \left\langle i \frac{\delta S_I}{\delta V_x} \frac{\delta S_I}{\delta V_y} + \frac{\delta^2 S_I}{\delta V_x \delta V_y} \right\rangle_{\text{1PI}}, \quad (33)$$

which is derived in Appendix A. Here the symbol 1PI means that in this expression it is necessary to keep only one-particle irreducible graphs, and S_I denotes the interaction. For convenience we denoted the argument of the effective action by the bold letter \mathbf{V} . (This equation can be also easily obtained using the background field method, which is not used in this paper.) Here

$$S_I = \frac{1}{4} \int d^8x \left(\phi^*(e^{2V} - 1)\phi + \tilde{\phi}^*(e^{-2V} - 1)\tilde{\phi} \right). \quad (34)$$

We substitute this expression into Eq. (33), taking into account the identity

$$\langle f(\phi, \phi^*) \rangle = \frac{1}{Z} f\left(\frac{1}{i} \frac{\delta}{\delta j}, \frac{1}{i} \frac{\delta}{\delta j^*}\right) Z, \quad (35)$$

where the generating functional Z is given by Eq. (14). After simple calculations we obtain⁵

$$\Gamma_{\mathbf{V}}^{(2)} = S_{\mathbf{V}}^{(2)} + S_{\text{gf}} + \left\langle -\frac{i}{2} \left(\text{Tr}(\mathbf{V}QJ_0\star) \right)^2 - i \text{Tr}(\mathbf{V}QJ_0\star \mathbf{V}QJ_0\star) - i \text{Tr}(\mathbf{V}^2J_0\star) \right\rangle + (PV), \quad (36)$$

where

$$\text{Tr} A = \text{tr} \int d^8x A_{xx}, \quad (37)$$

and tr denotes a usual matrix trace (if it is needed). (PV) denotes contributions of the Pauli–Villars fields,

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}; \quad J_0 = \begin{pmatrix} 0 & e^{2V} & 0 & 0 \\ e^{2V} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-2V} \\ 0 & 0 & e^{-2V} & 0 \end{pmatrix} P. \quad (38)$$

Due to the supersymmetric Ward identity the two-point Green function of the gauge superfield can be presented in the following form:

$$\Gamma_{\mathbf{V}}^{(2)} - S_{\text{gf}} = -\frac{1}{16\pi} \int \frac{d^4p}{(2\pi)^4} d^4\theta \mathbf{V}(\theta, -p) \partial^2 \Pi_{1/2} \mathbf{V}(\theta, p) d^{-1}(\alpha, \lambda, \mu/p), \quad (39)$$

where α is a renormalized coupling constant, and

$$\partial^2 \Pi_{1/2} = -\frac{1}{8} D^a \bar{D}^2 D_a \quad (40)$$

is a supersymmetric transversal projector. We will calculate the expression

$$\frac{d}{d \ln \Lambda} \left(d^{-1}(\alpha_0, \Lambda/p) - \alpha_0^{-1} \right) \Big|_{p=0} = -\frac{d\alpha_0^{-1}}{d \ln \Lambda} = \frac{\beta(\alpha_0)}{\alpha_0^2}. \quad (41)$$

(Here Λ and α are considered as independent variables.) From this equation it is evident that this expression is well defined. (Later we will demonstrate this in the three-loop approximation explicitly.) Note that here we implicitly use the higher derivative regularization, because it allows to perform differentiation with respect to $\ln \Lambda$ and set the external momentum p to 0.

⁵In order to find the contributions of ϕ - and $\tilde{\phi}$ -loops it is necessary to set $M = 0$.

5 Some useful tricks and summation of subdiagrams

In order to calculate expression (41) we consider

$$\frac{d}{d \ln \Lambda} \left(\Gamma_{\mathbf{V}}^{(2)} - S - S_{\text{gf}} \right) \Big|_{p=0}. \quad (42)$$

Making calculations in the limit $p \rightarrow 0$, where p is the external momentum, is possible, because the corresponding integrals are well defined in this limit. The higher derivative regularization and the differentiation with respect to $\ln \Lambda$ ensure that there are no IR divergences. This agrees with the results of Ref. [33] that the IR region does not affect to the β -function. In order to obtain a transversal part of the two-loop Green function of the gauge superfield by the simplest way, we make the substitution

$$\mathbf{V}(x, \theta) \rightarrow \bar{\theta}^a \bar{\theta}_a \theta^b \theta_b \equiv \theta^4, \quad (43)$$

so that

$$\int d^4 \theta \mathbf{V}(x, \theta) \partial^2 \Pi_{1/2} \mathbf{V}(x, \theta) \rightarrow -8. \quad (44)$$

(This is possible, because in the limit $p \rightarrow 0$ the gauge superfield \mathbf{V} does not depend on the coordinates x^μ .) In the momentum representation

$$\mathbf{V}(p, \theta) = \int d^4 x \mathbf{V}(x, \theta) e^{-ip_\alpha x^\alpha} \rightarrow (2\pi)^4 \delta^4(p) \theta^4. \quad (45)$$

Thus, after substitution (43) we obtain

$$\begin{aligned} (2\pi)^3 \delta^4(p) \frac{d}{d \ln \Lambda} \left(d^{-1}(\alpha_0, \Lambda/p) - \alpha_0^{-1} \right) \Big|_{p=0} &= (2\pi)^3 \delta^4(p) \frac{\beta(\alpha_0)}{\alpha_0^2} \Big|_{p=0} \\ &= \frac{d}{d \ln \Lambda} \left(\Gamma_{\mathbf{V}}^{(2)} - S - S_{\text{gf}} \right) \Big|_{p=0, \mathbf{V}(x, \theta) = \theta^4}. \end{aligned} \quad (46)$$

We will try to reduce the sum of Feynman diagrams for the considered theory to integrals of total derivatives. In the coordinate representation such an integral can be written as

$$\text{Tr} \left([x^\mu, \text{Something}] \right) = 0. \quad (47)$$

In order to find a β -function one should consider the massless theory. In the massless limit the fields ϕ and $\tilde{\phi}$ decouple. The Pauli–Villars contributions (for which this is not true) will be considered later. First, we will find a contribution of the field ϕ to the β -function. The contribution of the field $\tilde{\phi}$ can be found similarly. We will take it into account in the end.

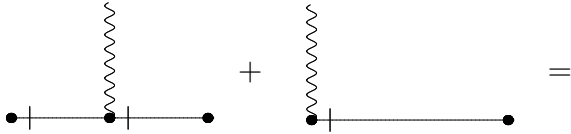
In order to extract commutators (47), we consider diagrams containing a vertex to that only one external line (and no internal lines) is attached. We can add such a diagram to a diagram, in which the external line is shifted to the nearest vertex. Let us formulate this rigorously. In the massless case

$$J_0 \rightarrow e^{2V} \frac{\bar{D}^2 D^2}{16\partial^2}; \quad I_0 \rightarrow (e^{2V} - 1) \frac{\bar{D}^2 D^2}{16\partial^2}; \quad P \rightarrow \frac{\bar{D}^2 D^2}{16\partial^2}; \quad \mathcal{V} \rightarrow (e^{2V} - 1), \quad (48)$$

so that $J_0 = I_0 + P$. As a consequence,

$$* \mathbf{V} J_0 = \frac{1}{1 - I_0} \mathbf{V}(I_0 + P) = \mathbf{V} P + \frac{1}{1 - I_0} (I_0 \mathbf{V} P + \mathbf{V} I_0) = \mathbf{V} P + * \mathcal{V} (P \mathbf{V} P + \mathbf{V} P). \quad (49)$$

The expression $\mathcal{V}(P \mathbf{V} P + \mathbf{V} P)$ corresponds to a sum of subdiagrams presented below. Making a substitution $\mathbf{V} \rightarrow \bar{\theta}^a \bar{\theta}_a \theta^b \theta_b$ we obtain



$$= (e^{2V} - 1) \left(-\theta^a \theta_a \bar{\theta}^b \frac{\bar{D}_b D^2}{4\partial^2} + \theta^a \theta_a \frac{D^2}{4\partial^2} + i\bar{\theta}^b (\gamma^\mu)_b{}^a \theta_a \frac{\bar{D}^2 D^2 \partial_\mu}{8\partial^4} - i\theta^a (\gamma^\mu)_a{}^b \bar{\theta}_b \frac{\bar{D}_b D^2 \partial_\mu}{4\partial^4} + \frac{\bar{D}^2 D^2}{16\partial^4} \right). \quad (50)$$

Only the first and the third terms give nontrivial contributions to the two-point function of the gauge superfield, because they contain $\bar{\theta}$. Really, finally it is necessary to obtain

$$\int d^4\theta \bar{\theta}^a \bar{\theta}_a \theta^b \theta_b,$$

while calculating a θ -part of a graph can not increase degrees of θ or $\bar{\theta}$. Therefore, we should have $\bar{\theta}^a \bar{\theta}_a$ from the beginning.

6 Reducing the sum of diagrams to integrals of total derivatives

6.1 One-loop approximation

For the general renormalizable $N = 1$ supersymmetric Yang-Mills theory, regularized by higher derivatives, the one-loop β -function was calculated in [24]. The result is given by an integral of a total derivative and agrees with the exact NSVZ β -function. Therefore, below we can make calculations starting from the two-loop approximation.

6.2 External V-lines are attached to different loops of the matter superfields

Let us try to find a sum of Feynman diagrams exactly to all orders of the perturbation theory. We will start with diagrams in that the external lines are attached to different

loops of matter superfields. Let us consider a loop of matter superfields with n vertexes. This loop is proportional to

$$\begin{aligned} & \text{Tr} \left\langle i\bar{\theta}^c(\gamma^\nu)_c{}^d \theta_d (e^{2V} - 1) \frac{\bar{D}^2 D^2 \partial_\nu}{8\partial^4} * -\theta^c \theta_c \bar{\theta}^d (e^{2V} - 1) \frac{\bar{D}_d D^2}{4\partial^2} * \right\rangle_n = \\ & = \frac{1}{n} \text{Tr} \left\langle i\bar{\theta}^c(\gamma^\nu)_c{}^d \theta_d (e^{2V} - 1) \frac{\bar{D}^2 D^2 \partial_\nu}{8\partial^2} *^2 -\theta^c \theta_c \bar{\theta}^d (e^{2V} - 1) \frac{\bar{D}_d D^2}{4\partial^2} *^2 \right\rangle_n. \end{aligned} \quad (51)$$

($*^2$ contains $n - 1$ vertexes, and one vertex corresponds to explicitly written $(e^{2V} - 1)$.) After simple algebraic transformations this expression can be written as

$$\begin{aligned} & \frac{1}{n} \text{Tr} \left\langle -\theta^c \theta_c \bar{\theta}^d * (e^{2V} - 1) \frac{\bar{D}_d D^2}{4\partial^2} * -\bar{\theta}^d \theta^c * (e^{2V} - 1) \frac{\bar{D}^2 D_c}{4\partial^2} * (e^{2V} - 1) \frac{\bar{D}_d D^2}{4\partial^2} * + \right. \\ & \left. + i\bar{\theta}^c(\gamma^\nu)_c{}^d \theta_d * (e^{2V} - 1) \frac{\bar{D}^2 D^2 \partial_\nu}{8\partial^4} * + \theta^2, \bar{\theta}^1, \theta^1, \theta^0 \text{ terms} \right\rangle_n = \\ & = \frac{1}{n} \text{Tr} \left\langle -2\theta^c \theta_c \bar{\theta}^d [\bar{\theta}_d, *] + i\bar{\theta}^c(\gamma^\mu)_c{}^d \theta_d [y_\mu^*, *] + \theta^2, \bar{\theta}^1, \theta^1, \theta^0 \text{ terms} \right\rangle_n, \end{aligned} \quad (52)$$

where $y_\nu^* = x_\nu - i\bar{\theta}^a(\gamma_\nu)_a{}^b \theta_b$ is an antichiral coordinate, such that

$$D_a y_\mu^* = 0. \quad (53)$$

Therefore, the considered loop with an external **V**-line can be written as

$$\text{Tr} \left\langle -2\theta^c \theta_c \bar{\theta}^d [\bar{\theta}_d, \ln *] + i\bar{\theta}^c(\gamma^\nu)_c{}^d \theta_d [y_\nu^*, \ln *] + \theta^2, \bar{\theta}^1, \theta^1, \theta^0 \text{ terms} \right\rangle. \quad (54)$$

Total derivatives in this expression give 0, because $\ln *$ does not contain ∂_μ/∂^4 . Thus, the loop gives only

$$\theta^2, \bar{\theta}^1, \theta^1, \theta^0 \text{ terms}. \quad (55)$$

The considered diagrams contain two loops of the matter superfields with an attached external **V**-line. Therefore, in order to calculate such diagrams it is necessary to multiply expressions (55), corresponding to each of these loops. As we explained above, in order to obtain a nontrivial result, it is necessary to have at least the second power of $\bar{\theta}$ and of θ . Therefore, all these terms vanish after the multiplication and subsequent calculation of the diagram. Thus, the sum of all such diagrams is given by an integral of a total derivative and is equal to 0.

6.3 External **V**-lines are attached to a single loop of the matter superfields

Now let us consider a case in that both external **V**-lines are attached to a single loop of the matter superfields. For simplicity we will consider only a contribution of the field ϕ . (This means that the external lines are attached to the ϕ loop. Other loops can certainly contain $\tilde{\phi}$ -propagators.) The matter is that in the massless limit the fields ϕ

and $\tilde{\phi}$ decouple. It is easy to see that a contribution of the field $\tilde{\phi}$ is exactly equal to the contribution of the field ϕ .

In the massless case we should calculate the diagrams

$$\begin{array}{ccc}
\text{Diagram 1} & \text{Diagram 2} & \text{Diagram 3}
\end{array} \quad (56)$$

where the vertexes are given by the corresponding terms in Eq. (50).

Let us shift θ -s to an arbitrary point of the loop, commuting them with matter propagators. This gives (the coefficients correspond to the expression (46); contribution of $\tilde{\phi}$ is not taken into account)⁶

$$\begin{aligned}
\text{Diagram 1} &= \frac{i}{64} \frac{d}{d \ln \Lambda} \text{Tr} \left\langle \theta^4 (e^{2V} - 1) \frac{\bar{D}^2 D^2 \partial^\mu}{\partial^4} * (e^{2V} - 1) \frac{\bar{D}^2 D^2 \partial_\mu}{\partial^4} * \right\rangle; \\
\text{Diagram 2} &= 2(\gamma^\mu)_d{}^c \frac{d}{d \ln \Lambda} \text{Tr} \left\langle \theta^4 \left((e^{2V} - 1) \frac{\bar{D}^2 D_c \partial_\mu}{16 \partial^4} * (e^{2V} - 1) \frac{\bar{D}^d D^2}{\partial^2} * \right. \right. \\
&\quad \left. \left. + (e^{2V} - 1) \frac{\bar{D}^2 D^2 \partial_\mu}{16 \partial^4} * (e^{2V} - 1) \frac{\bar{D}^2 D_c}{16 \partial^2} * (e^{2V} - 1) \frac{\bar{D}^d D^2}{\partial^2} * \right) \right\rangle; \\
\text{Diagram 3} &= -2i \frac{d}{d \ln \Lambda} \text{Tr} \left\langle \theta^4 \left(- (e^{2V} - 1) \frac{\bar{D}_d D^2}{4 \partial^2} * (e^{2V} - 1) \frac{\bar{D}^2}{8 \partial^2} * \right. \right. \\
&\quad \times (e^{2V} - 1) \frac{\bar{D}^d D^2}{4 \partial^2} * - (e^{2V} - 1) \frac{\bar{D}_d}{2 \partial^2} * (e^{2V} - 1) \frac{\bar{D}^d D^2}{4 \partial^2} * \\
&\quad \left. \left. + (e^{2V} - 1) \frac{\bar{D}_d D^c}{2 \partial^2} * (e^{2V} - 1) \frac{\bar{D}^2 D_c}{8 \partial^2} * (e^{2V} - 1) \frac{\bar{D}^d D^2}{4 \partial^2} * + (e^{2V} - 1) \frac{\bar{D}_d D^2}{4 \partial^2} * (e^{2V} - 1) \right. \right. \\
&\quad \left. \left. \times \frac{\bar{D}^2 D^c}{8 \partial^2} * (e^{2V} - 1) \frac{\bar{D}^2 D_c}{8 \partial^2} * (e^{2V} - 1) \frac{\bar{D}^d D^2}{4 \partial^2} * \right) \right\rangle.
\end{aligned}$$

We will start with the calculation of the following sum of diagrams:

$$\text{Diagram 1} + \frac{1}{2} \text{Diagram 2} \quad (57)$$

Using the identity

$$[x^\mu, \frac{\partial_\mu}{\partial^4}] = [-i \frac{\partial}{\partial p_\mu}, -\frac{i p^\mu}{p^4}] = -2\pi^2 \delta^4(p_E) = -2\pi^2 i \delta^4(p) \quad (58)$$

after simple algebraic transformations we obtain

$$\begin{aligned}
&2i \frac{d}{d \ln \Lambda} \text{Tr} \left\langle \theta^4 \left(\frac{i \pi^2}{8} * (e^{2V} - 1) \bar{D}^2 D^2 \delta^4(\partial_\alpha) + \left[y_\mu^*, (e^{2V} - 1) \frac{\bar{D}^2 D^2 \partial^\mu}{16 \partial^4} * \right] \right) \right\rangle \\
&= -\frac{d}{d \ln \Lambda} \text{Tr} \left\langle \frac{\pi^2}{4} \theta^4 * (e^{2V} - 1) \bar{D}^2 D^2 \delta^4(\partial_\alpha) \right\rangle.
\end{aligned} \quad (59)$$

⁶In order to obtain the contribution of a $\tilde{\phi}$ loop it is necessary to make a substitution $e^{2V} \rightarrow e^{-2V}$ and $*$ $\rightarrow \tilde{*}$.

Terms proportional to the δ -function will be calculated in the next section. (So far we have not yet found all such terms.) We will see that they give a part of the β -function proportional to the anomalous dimension.

Now let us calculate the diagrams

$$\begin{array}{c} \text{wavy line } \theta^a \theta_a \bar{\theta}^b \text{ --- } \text{shaded circle} \text{ --- } \text{wavy line } \theta^c \theta_c \bar{\theta}^d \end{array} + \frac{1}{2} \begin{array}{c} \text{wavy line } \theta^a \bar{\theta}^b \text{ --- } \text{shaded circle} \text{ --- } \text{wavy line } \theta^c \theta_c \bar{\theta}^d \end{array} \quad (60)$$

This sum can be written as

$$\begin{aligned} & \frac{d}{d \ln \Lambda} \text{Tr} \left\langle \theta^4 \left((\gamma^\mu)_d{}^c \left[y_\mu^*, (I_1)_c * (\bar{I}_1)^d * \right] + 2(\gamma^\mu)_d{}^c (I_0) \frac{\partial_\mu}{\partial^2} * \left((I_1)_c * (\bar{I}_1)^d * \right. \right. \right. \\ & + (\bar{I}_1)^d * (I_1)_c * \left. \left. \right) - 2i \left(2(I_2) * (\bar{I}_1)^d * (\bar{I}_1)_d * + 2(I_2)_d{}^c * (I_1)_c * (\bar{I}_1)^d * \right. \right. \\ & \left. \left. + 4(\bar{I}_1)_d * (I_1)^c * (I_1)_c * (\bar{I}_1)^d * \right) \right) \right\rangle + \text{terms proportional to a } \delta\text{-function.} \end{aligned} \quad (61)$$

(Terms proportional to a δ -function will be calculated later.)

In order to present this expression as an integral of a total derivative we will use the identity

$$\begin{aligned} & \text{Tr} \left(\theta^4 \left((\gamma^\mu)^{ab} [y_\mu^*, A] [\bar{\theta}_b, B] \{\theta_a, C\} + (\gamma_\mu)^{ab} (-1)^{P_A} \{\theta_a, B\} [\bar{\theta}_b, C] [y_\mu^*, A] \right. \right. \\ & \left. \left. - 4i [\theta^a, \{\theta_a, A\}] [\bar{\theta}^b, B] [\bar{\theta}_b, C] \right) \right) + \text{cyclic perm. of } A, B, C \\ & = \frac{1}{3} \text{Tr} \left(\theta^4 (\gamma^\mu)^{ab} \left[y_\mu^*, A [\bar{\theta}_b, B] \{\theta_a, C\} + (-1)^{P_A} \{\theta_a, B\} [\bar{\theta}_b, C] A \right] \right. \\ & \left. + \text{cyclic perm. of } A, B, C, \right) \end{aligned} \quad (62)$$

which was proved in Ref. [29]. For the completeness we also present this proof in the Appendix B. Here A , B , and C are arbitrary differential operators, constructed from the supersymmetric covariant derivatives, which do not explicitly depend on θ , and P_X is a Grassmannian parity of X .

Qualitative arguments presented in Ref. [29] allow to suggest that expression (61) for a diagram with n vertexes on the considered matter loop can be written in the following form:

$$\begin{aligned} & \frac{d}{d \ln \Lambda} \text{Tr} \left\langle \theta^4 (\gamma^\mu)_d{}^c \left[y_\mu^*, (I_1)_c * (\bar{I}_1)^d * \right] \right\rangle_n - \frac{6}{n(n+1)(n+2)} \frac{d}{d \ln \Lambda} \text{Tr} \left\langle \theta^4 \left(\right. \right. \\ & + (\gamma^\mu)^{ab} [y_\mu^*, *^3] [\bar{\theta}_b, *] [\theta_a, I_0] + (\gamma^\mu)^{ab} [\theta_a, *^3] [\bar{\theta}_b, *] [y_\mu^*, I_0] - 4i \{ \theta^a, [\theta_a, *^3] \} [\bar{\theta}^b, *] [\bar{\theta}_b, I_0] \\ & + (\gamma^\mu)^{ab} [y_\mu^*, *] [\bar{\theta}_b, I_0] [\theta_a, *^3] + (\gamma^\mu)^{ab} [\theta_a, *] [\bar{\theta}_b, I_0] [y_\mu^*, *^3] - 4i \{ \theta^a, [\theta_a, *] \} [\bar{\theta}^b, I_0] [\bar{\theta}_b, *^3] \\ & + (\gamma^\mu)^{ab} [y_\mu^*, I_0] [\bar{\theta}_b, *^3] [\theta_a, *] + (\gamma^\mu)^{ab} [\theta_a, I_0] [\bar{\theta}_b, *^3] [y_\mu^*, *] - 4i \{ \theta^a, [\theta_a, I_0] \} [\bar{\theta}^b, *^3] [\bar{\theta}_b, *] \\ & \left. \left. + (\gamma^\mu)^{ab} [y_\mu^*, *] [\bar{\theta}_b, *] [\theta_a, *] + (\gamma^\mu)^{ab} [\theta_a, *] [\bar{\theta}_b, *] [y_\mu^*, *] - 4i \{ \theta^a, [\theta_a, *] \} [\bar{\theta}^b, *] [\bar{\theta}_b, *] \right) \right\rangle_n \end{aligned} \quad (63)$$

In order to prove this, it is necessary to calculate all commutators and take into account Eq. (28). For example, if $n = a + b + c + 3$, then

$$\begin{aligned}
A(*)_a B(*)_b C(*)_c &= \frac{6}{n(n+1)(n+2)} \left(A(*^4)_a B(*)_b C(*)_c + A(*)_a B(*^4)_b C(*)_c \right. \\
&+ A(*)_a B(*)_b C(*^4)_c + A(*^3)_a B(*^2)_b C(*)_c + A(*^2)_a B(*^3)_b C(*)_c + A(*^3)_a B(*)_b C(*^2)_c \\
&\left. + A(*^2)_a B(*)_b C(*^3)_c + A(*)_a B(*^2)_b C(*^3)_c + A(*)_a B(*^3)_b C(*^2)_c + A(*^2)_a B(*^2)_b C(*^2)_c \right), \tag{64}
\end{aligned}$$

because

$$\begin{aligned}
&\frac{1}{6}(a+b+c+3)(a+b+c+4)(a+b+c+5) = \frac{1}{6}(a+1)(a+2)(a+3) \\
&+ \frac{1}{6}(b+1)(b+2)(b+3) + \frac{1}{6}(c+1)(c+2)(c+3) + \frac{1}{2}(a+1)(a+2)(b+1) \\
&+ \frac{1}{2}(b+1)(b+2)(a+1) + \frac{1}{2}(a+1)(a+2)(c+1) + \frac{1}{2}(c+1)(c+2)(a+1) \\
&+ \frac{1}{2}(c+1)(c+2)(b+1) + \frac{1}{2}(b+1)(b+2)(c+1) + (a+1)(b+1)(c+1). \tag{65}
\end{aligned}$$

(A similar, but larger identity can be written for the term with four $*$ in Eq. (61).)

Applying identity (62) to (63), we present expression (61) as an integral of a total derivative:

$$\begin{aligned}
&\frac{d}{d \ln \Lambda} \text{Tr} \left\langle \theta^4 (\gamma^\mu)_d^c \left[y_\mu^*, (I_1)_c * (\bar{I}_1)^d * \right] \right\rangle_n - \frac{2}{n(n+1)(n+2)} \frac{d}{d \ln \Lambda} \text{Tr} \left\langle \theta^4 (\gamma^\mu)^{ab} \right. \\
&\times \left[y_\mu^*, *^3 [\bar{\theta}_b, *][\theta_a, I_0] + [\theta_a, *^3][\bar{\theta}_b, *] I_0 + *[\bar{\theta}_b, I_0][\theta_a, *^3] + [\theta_a, *][\bar{\theta}_b, I_0] *^3 \right. \\
&\left. + I_0[\bar{\theta}_b, *^3][\theta_a, *] + [\theta_a, I_0][\bar{\theta}_b, *^3] * + *[\bar{\theta}_b, *][\theta_a, *] + [\theta_a, *][\bar{\theta}_b, *] * \right] \right\rangle_n \\
&\quad + \text{terms, proportional to a } \delta\text{-function.} \tag{66}
\end{aligned}$$

Calculating the commutators with θ and $\bar{\theta}$ we obtain

$$\begin{aligned}
&\frac{d}{d \ln \Lambda} \text{Tr} \left\langle \theta^4 (\gamma^\mu)_d^c \left[y_\mu^*, (I_1)_c * (\bar{I}_1)^d * \right] \right\rangle_n - \frac{2(\gamma^\mu)^{ab}}{n(n+1)(n+2)} \text{Tr} \frac{d}{d \ln \Lambda} \left\langle \theta^4 \left[y_\mu^*, *^4 (\bar{I}_1)_b * \right. \right. \\
&\times (I_1)_a + \left(*^3 (I_1)_a * + *^2 (I_1)_a *^2 + * (I_1)_a *^3 \right) * (\bar{I}_1)_b * I_0 + * (\bar{I}_1)_b \left(*^3 (I_1)_a * + *^2 \right. \\
&\times (I_1)_a *^2 + * (I_1)_a *^3 \left. \right) + * (I_1)_a * (\bar{I}_1)_b *^3 + I_0 \left(*^3 (\bar{I}_1)_b * + *^2 (\bar{I}_1)_b *^2 + * (\bar{I}_1)_b *^3 \right) \\
&\times * (I_1)_a * + (I_1)_a \left(*^3 (\bar{I}_1)_b * + *^2 (\bar{I}_1)_b *^2 + * (\bar{I}_1)_b *^3 \right) * + *^2 (\bar{I}_1)_b *^2 (I_1)_a * + * (I_1)_a \\
&\left. \times *^2 (\bar{I}_1)_b *^2 \right] \right\rangle_n + \text{terms, proportional to a } \delta\text{-function.} \tag{67}
\end{aligned}$$

Thus, the sum of all remaining diagrams is also given by an integral of a total derivative.

In order to simplify the obtained expressions we derive an identity, which corresponds to shifting a loop momentum in an integral of a total derivative. For this purpose let us formally assume that y_μ^* and θ^4 do not commute. Then due to the Jacobi identity

$$[[\theta^4, y_\mu^*], A] = [\theta^4, [y_\mu^*, A]] - [y_\mu^*, [\theta^4, A]]. \quad (68)$$

As earlier, we assume that A is a differential operator constructed from the supersymmetric covariant derivatives. As a consequence

$$[y_\mu^*, A] = -2i(\gamma_\mu)^{ab}\theta_a[\bar{\theta}_b, A] + O(\theta^0). \quad (69)$$

Therefore,

$$[[\theta^4, y_\mu^*], A] = -2i(\gamma_\mu)^{ab}[\theta^4, \theta_a[\bar{\theta}_b, A]] + 2i(\gamma_\mu)^{ab}\theta_a[\bar{\theta}_b, [\theta^4, A]] + O(\theta^3) = O(\theta^3). \quad (70)$$

(Terms that do not contain θ^4 vanish after integration over $d^4\theta$.) So, without using the relation $[y_\mu^*, \theta^4] = 0$ we obtained

$$[[\theta^4, y_\mu^*], A] = O(\theta^3). \quad (71)$$

Because the operation Tr includes the integration over $d^4\theta$, this means that it is possible to make cyclic permutations ($P_A = P_B$)

$$\text{Tr}\langle\theta^4[y_\mu^*, AB]\rangle = \text{Tr}\langle[\theta^4, y_\mu^*]AB\rangle = \text{Tr}\langle A[\theta^4, y_\mu^*]B\rangle = (-1)^{P_A}\text{Tr}\langle\theta^4[y_\mu^*, BA]\rangle. \quad (72)$$

Actually this corresponds to shifts of the loop momentum in an integral of a total derivative. Because the integrals are well defined, such shifts do not change the integral.

Taking into account the possibility of making such cyclic permutations, one can simplify expression (67). In Appendix C we prove that it can be written as

$$\begin{aligned} & - \sum_{a+b+2=n} \frac{2(b+1)(\gamma^\mu)^{cd}}{n} \frac{d}{d\ln\Lambda} \text{Tr}\langle\theta^4[y_\mu^*, (I_1)_c(*)_a(\bar{I}_1)_d(*)_b]\rangle \\ & + \text{terms proportional to a } \delta\text{-function.} \end{aligned} \quad (73)$$

Collecting all results we obtain the following expression for a ϕ -contribution to (46):

$$\begin{aligned} & \frac{d}{d\ln\Lambda} \text{Tr}\langle\theta^4[y_\mu^*, 2i(e^{2V} - 1)\frac{\bar{D}^2 D^2 \partial_\mu}{16\partial^4} * - \sum_{a+b+2=n} \frac{2(b+1)(\gamma^\mu)^{cd}}{n} (I_1)_c(*)_a(\bar{I}_1)_d(*)_b]\rangle_n \\ & + \text{terms proportional to a } \delta\text{-function.} \end{aligned} \quad (74)$$

The terms written explicitly are certainly equal to 0. The exact NSVZ β -function is obtained from the terms proportional to a δ -function, which are calculated in the next section.

Taking into account a possibility of making cyclic permutations in the expression which is commuted with y_μ^* , (74) can be rewritten as

$$\begin{aligned} & \frac{d}{d \ln \Lambda} \frac{1}{n} \text{Tr} \left\langle \theta^4 \left[y_\mu^*, * 2i(e^{2V} - 1) \frac{\bar{D}^2 D^2 \partial_\mu}{16 \partial^4} * - 2(\gamma^\mu)^{cd} * (I_1)_c * (\bar{I}_1)_d * \right] \right\rangle_n \\ & + \text{terms proportional to a } \delta\text{-function.} \end{aligned} \quad (75)$$

It is easy to see that this expression can be presented in the form

$$\frac{d}{d \ln \Lambda} \frac{i}{n} \text{Tr} \left\langle \theta^4 \left[(y^\mu)^*, [y_\mu^*, *] \right] \right\rangle_n + \text{terms proportional to a } \delta\text{-function.} \quad (76)$$

Thus, (taking into account contribution of $\tilde{\phi}$ -loops) finally we obtain

$$i \frac{d}{d \ln \Lambda} \text{Tr} \left\langle \theta^4 \left[(y^\mu)^*, [y_\mu^*, \ln(**)] \right] \right\rangle + \text{terms proportional to a } \delta\text{-function.} \quad (77)$$

7 Derivation of the NSVZ β -function

In the previous section it was found that all integrals giving the β -function are integrals of total derivatives. However, they are not equal to 0, because

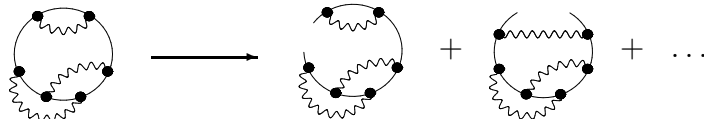
$$\frac{1}{\pi^2} \int d^4 q \frac{1}{q^2} \frac{d}{dq^2} f(q^2) = \int_0^\infty dq^2 \frac{d}{dq^2} f(q^2) = f(\infty) - f(0) = -f(0) \neq 0. \quad (78)$$

($f(\infty) = 0$ due to the higher derivative regularization.) This is equivalent to taking into account terms with a δ -function. Really, let us rewrite this equality as follows:

$$\begin{aligned} \int_0^\infty dq^2 \frac{d}{dq^2} f(q^2) &= \frac{1}{2\pi^2} \int d^4 q \frac{q^\mu}{q^4} \frac{\partial f}{\partial q^\mu} = \frac{1}{2\pi^2} \int d^4 q \left(\frac{\partial}{\partial q^\mu} \left(\frac{q^\mu f}{q^4} \right) - f \frac{\partial}{\partial q^\mu} \left(\frac{q^\mu}{q^4} \right) \right) \\ &= - \int d^4 q \delta^4(q) f = -f(0). \end{aligned} \quad (79)$$

Thus, we see that total derivatives with respect to q^2 are equivalent to total derivatives with respect to q^μ plus terms proportional to $\delta^4(q)$. In the approach described in the previous section δ -functions appear, if y_μ^* is commuted with ∂_μ/∂^4 as in Eq. (58). In this section we calculate all such terms.

Qualitatively, the δ -function allows to perform an integration over a momentum of the considered matter loop. This corresponds to cutting the loop, which gives diagrams for the two-point Green function of the matter superfield [22]. For example,



Let us derive this by a rigorous method. First we consider expression (59). Omitting total derivatives (corresponding to $\partial/\partial q^\mu$) we obtain

$$\begin{aligned}
& -\frac{d}{d \ln \Lambda} \text{Tr} \left\langle \frac{\pi^2}{4} \theta^4 * (e^{2V} - 1) \bar{D}^2 D^2 \delta^4(\partial) \right\rangle = \\
& = -\frac{d}{d \ln \Lambda} \int d^8 x d^8 y \delta_{xy}^8 \left\langle \frac{\pi^2}{4} \theta^4 * (e^{2V} - 1) \left(-\frac{\bar{D}^2 D^2}{16 \partial^2} \right) \bar{D}^2 D^2 \delta^4(\partial) \delta_{xy}^8 \right\rangle = \\
& = -\frac{d}{d \ln \Lambda} \int d^8 x d^8 y \delta_{xy}^8 \left\langle \frac{\pi^2}{4} \theta^4 (1 - *) \bar{D}^2 D^2 \delta^4(\partial) \delta_{xy}^8 \right\rangle = \\
& = \frac{d}{d \ln \Lambda} \int d^8 x d^8 y \delta_{xy}^8 \left\langle \frac{\pi^2}{4} \theta^4 * \bar{D}^2 D^2 \delta^4(\partial) \delta_{xy}^8 \right\rangle. \quad (80)
\end{aligned}$$

Taking into account that

$$\delta^4(\partial) \delta^4(x - y) = \int \frac{d^4 q}{(2\pi)^4} \delta^4(q) e^{-iq_\alpha(x^\alpha - y^\alpha)} = \frac{1}{(2\pi)^4} \quad (81)$$

and calculating θ -integrals, we obtain

$$-\frac{d}{d \ln \Lambda} \text{Tr} \left\langle \frac{\pi^2}{4} \theta^4 * (e^{2V} - 1) \bar{D}^2 D^2 \delta^4(\partial) \right\rangle = \frac{\pi^2}{(2\pi)^4} \frac{d}{d \ln \Lambda} \int d^4 x d^4 y \left\langle * \bar{D}_x^2 D_x^2 \delta_{xy}^8 \right\rangle \Big|_{\theta_x = \theta_y}. \quad (82)$$

Using Eq. (32) for the function G^{-1} this can be presented in the form

$$\begin{aligned}
& \frac{\pi^2}{(2\pi)^4} \frac{d}{d \ln \Lambda} \int d^4 x d^4 y G^{-1} \bar{D}_x^2 D_x^2 \delta_{xy}^8 \Big|_{\theta_x = \theta_y} = \frac{4\pi^2}{(2\pi)^4} \frac{d}{d \ln \Lambda} \int d^4 x d^4 y G^{-1} \delta^4(x - y) = \\
& = 4\pi^2 \frac{d}{d \ln \Lambda} G^{-1} \delta^4(p) \Big|_{p=0}. \quad (83)
\end{aligned}$$

This expression is not well defined. Thus, it is written formally. However, we will see that after adding the other contributions a well defined result is obtained.

δ -functions are also present in expression (73) (or (61)), if the matter loop contains coinciding momentums. Really, taking into account that

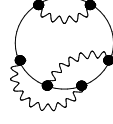
$$(I_1)_c = (e^{2V} - 1) \frac{\bar{D}^2 D_c}{8 \partial^2}; \quad (\bar{I}_1)_d = (e^{2V} - 1) \frac{\bar{D}_d D^2}{8 \partial^2}, \quad (84)$$

$\partial_\mu / \partial^4$ appears due to the following identities (It is assumed that momentums in I_1 and \bar{I}_1 coincide):

$$(I_1)_c \cdot (\bar{I}_1)_d \rightarrow \frac{\bar{D}^2 D_c}{8 \partial^2} \cdot \frac{\bar{D}_d D^2}{8 \partial^2} = \frac{i}{2} ((1 + \gamma_5) \gamma^\mu)_{cd} \frac{\partial_\mu}{32 \partial^4} \bar{D}^2 D^2; \quad (85)$$

$$(\bar{I}_1)_d \cdot (I_1)_c \rightarrow \frac{\bar{D}_d D^2}{8 \partial^2} \cdot \frac{\bar{D}^2 D_c}{8 \partial^2} = -\frac{i}{2} ((1 + \gamma_5) \gamma^\mu)_{cd} \frac{\partial_\mu}{32 \partial^4} D^a \bar{D}^2 D_a. \quad (86)$$

Two (or more) momentums coincide, if two cuts of the matter loop make a diagram disconnected. An example of such a diagram is



For analyzing such diagrams we will use the identities

$$\begin{aligned}
(\bar{I}_1)_b \cdot I_0 &\rightarrow \frac{\bar{D}_b D^2}{8\partial^2} \cdot \frac{\bar{D}^2 D^2}{16\partial^2} = -\frac{\bar{D}_b D^2}{8\partial^2}; & I_0 \cdot (I_1)_a &\rightarrow \frac{\bar{D}^2 D^2}{16\partial^2} \cdot \frac{\bar{D}^2 D_a}{8\partial^2} = -\frac{\bar{D}^2 D_a}{8\partial^2}; \\
I_0 \cdot (\bar{I}_1)_b &\rightarrow \frac{\bar{D}^2 D^2}{16\partial^2} \cdot \frac{\bar{D}_b D^2}{8\partial^2} = 0; & (I_1)_a \cdot I_0 &\rightarrow \frac{\bar{D}^2 D_a}{8\partial^2} \cdot \frac{\bar{D}^2 D^2}{16\partial^2} = 0; \\
I_0 \cdot I_0 &\rightarrow \frac{\bar{D}^2 D^2}{16\partial^2} \cdot \frac{\bar{D}^2 D^2}{16\partial^2} = -\frac{\bar{D}^2 D^2}{16\partial^2}.
\end{aligned} \tag{87}$$

Let us assume that there are p coinciding momenta q in the considered matter loop (to that the external lines are attached). Then the corresponding diagram contributing to the (connected) two-point Green function of the matter superfield is not 1PI and consists of p parts, connected by a single line of the matter superfield. (If there are several groups of coinciding momenta, each group should be considered separately.)

Let us assume that the parts of such a diagram contains c_i ($i = 1, \dots, p$) vertexes on the matter line (to that the external lines are attached).⁷ We will denote expressions for these parts by G_1, G_2, \dots, G_p . Due to identities (87) the following variants are possible:

$$\begin{aligned}
a+1 &= c_1; & b+1 &= c_2 + c_3 + \dots + c_p; \\
a+1 &= c_2; & b+1 &= c_1 + c_3 + \dots + c_p; \\
&\dots & & \\
a+1 &= c_p; & b+1 &= c_1 + c_2 + \dots + c_{p-1}, \\
\text{where} & & n &= c_1 + c_2 + \dots + c_p,
\end{aligned} \tag{88}$$

because terms with (I_1) -s give a nontrivial result only if there are no lines with the momentum q between $(I_1)_a$ and $(\bar{I}_1)_b$. (We assume that momenta in (I_1) and (\bar{I}_1) are equal to q .) However, any number of such lines can be between $(\bar{I}_1)_b$ and $(I_1)_a$. According to the results of the previous section, it is necessary to calculate (and subtract) a singular part of the expression

$$\frac{d}{d \ln \Lambda} \text{Tr} \left\langle \theta^4 \left[y_\mu^*, 2i(e^{2V} - 1) \frac{\bar{D}^2 D^2 \partial_\mu}{16\partial^4} * - \sum_{a+b+2=n} \frac{2(b+1)(\gamma^\mu)^{cd}}{n} (I_1)_c(*)_a (\bar{I}_1)_d(*)_b \right] \right\rangle. \tag{89}$$

A singular part of the first term has been already found and is given by (83) (with the opposite sign). For a diagram that contains a sequence of subdiagrams G_1, G_2, \dots, G_p it can be written as

⁷For the diagram, presented above, $p = 2$, $c_1 = 2$, and $c_2 = 4$.

$$\begin{aligned}
& (-1)^{p-1} p \cdot \frac{d}{d \ln \Lambda} \text{Tr} \left(\theta^4 \left[y_\mu^*, 2i \frac{\bar{D}^2 D^2 \partial_\mu}{16 \partial^4} \right]_{\text{Singular part}} G_1 G_2 \dots G_p \right) \\
& = (-1)^{p-1} p \cdot \frac{\pi^2}{4} \frac{d}{d \ln \Lambda} \text{Tr} \left(\theta^4 G_1 G_2 \dots G_p \bar{D}^2 D^2 \delta^4(q) \right). \quad (90)
\end{aligned}$$

The factor p is present, because there are p variants by which $\bar{D}^2 D^2 \partial_\mu / 16 \partial^4$ can be placed between G_i .

In order to calculate a singular part of the second term, we consider

$$-2(\gamma^\mu)^{cd} \frac{d}{d \ln \Lambda} \sum_{a+b+2=n} \frac{(b+1)}{n} \text{Tr} \left\langle \theta^4 \left[y_\mu^*, (I_1)_c(*)_a (\bar{I}_1)_d(*)_b \right]_{\text{Singular part}} \right\rangle. \quad (91)$$

Using a possibility of making cyclic permutations inside the commutator and identity (86), this expression can be written as

$$\begin{aligned}
& (-1)^{p-2} \frac{d}{d \ln \Lambda} \text{Tr} \left\langle \theta^4 \left[y_\mu^*, \frac{i \partial^\mu}{8 \partial^4} G_1 G_2 \dots G_p \bar{D}^a D^2 D_a \right]_{\text{Singular part}} \right\rangle \\
& \times \left(\frac{c_1 + \dots + c_{p-1}}{c_1 + c_2 + \dots c_n} + \frac{c_1 + \dots + c_{p-2} + c_p}{c_1 + c_2 + \dots c_n} + \dots + \frac{c_2 + \dots + c_p}{c_1 + c_2 + \dots c_n} \right) \\
& = (-1)^p (p-1) \cdot \frac{\pi^2}{4} \frac{d}{d \ln \Lambda} \text{Tr} \left(\theta^4 G_1 G_2 \dots G_p \bar{D}^2 D^2 \delta^4(q) \right), \quad (92)
\end{aligned}$$

because

$$D^a \bar{D}^2 D_a \delta^4(\partial) = \bar{D}^2 D^2 \delta^4(\partial). \quad (93)$$

Therefore, a ratio of the coefficients in the first and the second terms of Eq. (89) is

$$-\frac{p-1}{p}. \quad (94)$$

Taking a sum of both contributions we see that a coefficient is proportional to

$$1 - \frac{p-1}{p} = \frac{1}{p}. \quad (95)$$

The coefficient 1 corresponds to the expansion of G^{-1} (see Eq. (83)). Therefore, taking into account the expansions

$$\ln(1-x) = - \sum_{p=1}^{\infty} \frac{x^p}{p}; \quad \frac{1}{1-x} = \sum_{p=0}^{\infty} x^p, \quad (96)$$

we see that singular parts of the commutators give the contribution to Eq. (46)

$$-4\pi^2 \delta^4(p) \Big|_{p=0} \frac{d \ln G}{d \ln \Lambda} = -4\pi^2 \delta^4(p) \Big|_{p=0} \frac{d}{d \ln \Lambda} \left(\ln(ZG) - \ln Z \right) = -4\pi^2 \delta^4(p) \Big|_{p=0} \gamma(\alpha_0). \quad (97)$$

Here the expression $\gamma(\alpha_0)$ is well defined, unlike the corresponding expression in Eq. (83). Thus, after taking into account all contributions, the well defined result is obtained.

Diagrams with a loop of $\tilde{\phi}$ -fields (to that external lines are attached) give exactly the same result. Therefore, due to Eq. (46) a β -function is given by the sum of the one-loop contribution α^2/π and

$$\Delta\beta = -\frac{\alpha^2}{\pi}\gamma(\alpha). \quad (98)$$

Thus, we obtain the exact NSVZ β -function

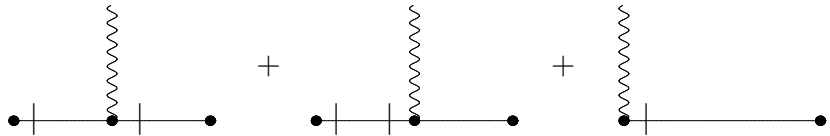
$$\beta(\alpha) = \frac{\alpha^2}{\pi}(1 - \gamma(\alpha)). \quad (99)$$

8 Pauli–Villars contributions

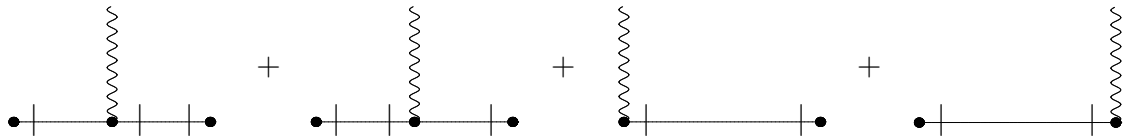
Previous calculation was formal, because so far we did not take into account contributions of the Pauli–Villars fields. However, these contributions can be considered in a similar way.

8.1 Summation of subdiagrams

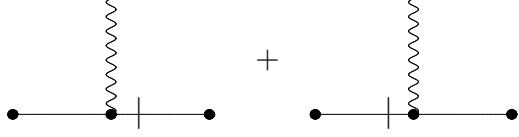
Let us start with the summation of subdiagrams. For the Pauli–Villars fields there are four different types of subdiagrams. Below we will calculate elements 22, 23, 14 and 11 of the corresponding matrix. After the substitution $V \rightarrow \theta^4$ and some algebraic transformations (omitting for simplicity expressions for the left vertexes) they can be presented in the following form:



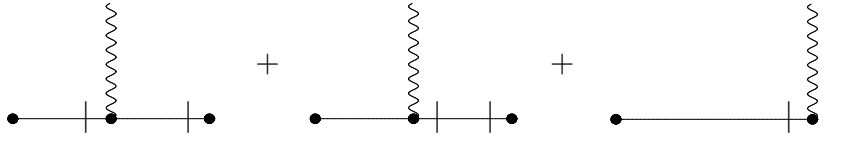
$$\begin{aligned}
&= i\bar{\theta}^a(\gamma_\mu)_a{}^b\theta_b \frac{\bar{D}^2 D^2 \partial_\mu}{8(\partial^2 + M^2)^2} - \theta^a \theta_a \bar{\theta}^b \frac{\bar{D}_b D^2}{4(\partial^2 + M^2)} + \text{terms without } \bar{\theta} \\
&= i\bar{\theta}^a(\gamma_\mu)_a{}^b\theta_b \left[y_\mu^*, \frac{\bar{D}^2 D^2}{16(\partial^2 + M^2)} \right] - 2\theta^a \theta_a \bar{\theta}^b \left[\bar{\theta}_b, \frac{\bar{D}^2 D^2}{16(\partial^2 + M^2)} \right] + \text{terms without } \bar{\theta}.
\end{aligned} \quad (100)$$



$$\begin{aligned}
&= i\bar{\theta}^b(\gamma_\mu)_a{}^b\theta_b \frac{M\bar{D}^2 \partial_\mu}{2(\partial^2 + M^2)^2} - \theta^a \theta_a \bar{\theta}^b \frac{M\bar{D}_b}{\partial^2 + M^2} + \text{terms without } \bar{\theta} \\
&= i\bar{\theta}^a(\gamma_\mu)_a{}^b\theta_b \left[y_\mu^*, \frac{M\bar{D}^2}{4(\partial^2 + M^2)} \right] - 2\theta^a \theta_a \bar{\theta}^b \left[\bar{\theta}_b, \frac{M\bar{D}^2}{4(\partial^2 + M^2)} \right] + \text{terms without } \bar{\theta}.
\end{aligned} \quad (101)$$



$$\begin{aligned}
&= -i\bar{\theta}^b(\gamma_\mu)_a{}^b\theta_a \frac{MD^2\partial_\mu}{2(\partial^2 + M^2)^2} - \bar{\theta}^b \frac{M\bar{D}_b D^2}{4(\partial^2 + M^2)^2} + \text{terms without } \bar{\theta} \\
&= -i\bar{\theta}^a(\gamma_\mu)_a{}^b\theta_b \left[y_\mu^*, \frac{MD^2}{4(\partial^2 + M^2)} \right] + 2\theta^a\theta_a\bar{\theta}^b \left[\bar{\theta}_b, \frac{MD^2}{4(\partial^2 + M^2)} \right] \\
&\quad - \bar{\theta}^b \frac{M\bar{D}_b D^2}{4(\partial^2 + M^2)^2} + \text{terms without } \bar{\theta}.
\end{aligned} \tag{102}$$



$$\begin{aligned}
&= -i\bar{\theta}^b(\gamma_\mu)_a{}^b\theta_a \frac{D^2\bar{D}^2\partial_\mu}{8(\partial^2 + M^2)^2} - \theta^a\bar{\theta}^b \frac{D_a\bar{D}_b}{\partial^2 + M^2} + \theta^a\theta_a\bar{\theta}^b \frac{D^2\bar{D}_b}{4(\partial^2 + M^2)} \\
&\quad - \bar{\theta}^b \frac{\bar{D}_b D^2\bar{D}^2}{16(\partial^2 + M^2)^2} - \bar{\theta}^b \frac{\bar{D}_b}{\partial^2 + M^2} + \text{terms without } \bar{\theta} = \\
&= -i\bar{\theta}^a(\gamma_\mu)_a{}^b\theta_b \left[y_\mu^*, \frac{D^2\bar{D}^2}{16(\partial^2 + M^2)} \right] + 2\theta^a\theta_a\bar{\theta}^b \left[\bar{\theta}_b, \frac{D^2\bar{D}^2}{16(\partial^2 + M^2)} \right] \\
&\quad - \bar{\theta}^b \frac{\bar{D}_b D^2\bar{D}^2}{16(\partial^2 + M^2)^2} - \bar{\theta}^b \frac{\bar{D}_b}{\partial^2 + M^2} + \text{terms without } \bar{\theta}.
\end{aligned} \tag{103}$$

The other matrix elements are calculated similarly. The whole matrix corresponding to sums of subdiagrams (100) — (103) is written as

$$\begin{aligned}
&\tilde{Q} \left(\bar{\theta}^a \mathcal{V} \times \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{i(\gamma^\mu)_a{}^b D_b \bar{D}^2 \partial_\mu}{4(\partial^2 + M^2)^2} + \frac{\bar{D}_a}{\partial^2 + M^2} & 0 & 0 & \frac{M\bar{D}_a D^2}{4(\partial^2 + M^2)^2} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{M\bar{D}_a D^2}{4(\partial^2 + M^2)^2} & \frac{i(\gamma^\mu)_a{}^b D_b \bar{D}^2 \partial_\mu}{4(\partial^2 + M^2)^2} + \frac{\bar{D}_a}{\partial^2 + M^2} & 0 \end{pmatrix} \right. \\
&\quad \left. + i\bar{\theta}^a(\gamma^\mu)_a{}^b\theta_b [y_\mu^*, \tilde{Q}I_0] - 2\theta^a\theta_a\bar{\theta}^b [\bar{\theta}_b, \tilde{Q}I_0] + \text{terms without } \bar{\theta} \right), \tag{104}
\end{aligned}$$

where

$$\tilde{Q} \equiv \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{105}$$

satisfies the identities

$$[\tilde{Q}, I_0] = 0; \quad [\tilde{Q}, \star] = 0; \quad \tilde{Q}^2 = 1. \quad (106)$$

I_0 is defined by Eq. (20), and \mathcal{V} (corresponding to the left vertexes) is given by (21). Terms without $\bar{\theta}$ do not contribute to diagrams for the β -function. (The integral over $d^4\theta$ is nontrivial only if a diagram contains θ^4 and, in particular, $\bar{\theta}^2$.)

8.2 External lines are attached to different matter loops

This case is very similar to the massless one. A loop of matter superfields to that an external line is attached is now proportional to (for simplicity we omit $-\sum_I c_I$)

$$\begin{aligned} & \text{Tr} \left\langle i\bar{\theta}^c (\gamma^\mu)_c{}^d \theta_d [y_\mu^*, \tilde{Q} I_0] \star - 2\theta^c \theta_c \bar{\theta}^d [\bar{\theta}_d, \tilde{Q} I_0] \star + \bar{\theta}^1 \text{ terms} \right\rangle_n = \\ & = \frac{1}{n} \text{Tr} \left\langle i\bar{\theta}^c (\gamma^\mu)_c{}^d \theta_d [y_\mu^*, \tilde{Q} I_0] \star^2 - 2\theta^c \theta_c \bar{\theta}^d [\bar{\theta}_d, \tilde{Q} I_0] \star^2 + \bar{\theta}^1 \text{ terms} \right\rangle_n. \end{aligned} \quad (107)$$

After some simple (but nontrivial) algebraic transformations this expression can be rewritten as

$$\begin{aligned} & \frac{1}{n} \text{Tr} \left\langle \tilde{Q} \left(-2\theta^c \theta_c \bar{\theta}^d \star [\bar{\theta}_d, I_0] \star + i\bar{\theta}^c (\gamma^\nu)_c{}^d \theta_d \star [y_\mu^*, I_0] \star \right) + \theta^2, \bar{\theta}^1, \theta^1, \theta^0 \text{ terms} \right\rangle_n = \\ & = \frac{1}{n} \text{Tr} \left\langle \tilde{Q} \left(-2\theta^c \theta_c \bar{\theta}^d [\bar{\theta}_d, \star] + i\bar{\theta}^c (\gamma^\mu)_c{}^d \theta_d [y_\mu^*, \star] \right) + \theta^2, \bar{\theta}^1, \theta^1, \theta^0 \text{ terms} \right\rangle_n. \end{aligned} \quad (108)$$

Therefore, as earlier, a matter loop is given by

$$\text{Tr} \left\langle \tilde{Q} \left(-2\theta^c \theta_c \bar{\theta}^d [\bar{\theta}_d, \ln \star] + i\bar{\theta}^c (\gamma^\nu)_c{}^d \theta_d [y_\nu^*, \ln \star] \right) + \theta^2, \bar{\theta}^1, \theta^1, \theta^0 \text{ terms} \right\rangle. \quad (109)$$

As in the massless case, multiplying expressions for two such loops we obtain that all diagrams in that external lines are attached to different matter loops are given by integrals of total derivatives. All these integrals are evidently equal to 0.

8.3 External lines are attached to a single matter loop

Calculation of such diagrams in the massive case has some differences from the massless case. We construct Feynman rules in the massive case using Eq. (104). Note that they are different from the corresponding rules in the massless case, because the expression

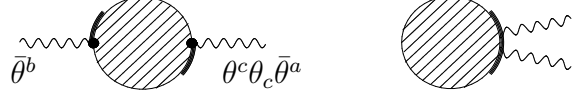
$$i\bar{\theta}^a (\gamma^\mu)_a{}^b \theta_b [y_\mu^*, I_0] \quad (110)$$

in the massless case gives

$$i\bar{\theta}^a (\gamma^\mu)_a{}^b \theta_b \left[y_\mu^*, \frac{\bar{D}^2 D^2}{16\partial^2} \right] = i\bar{\theta}^a (\gamma^\mu)_a{}^b \theta_b \frac{\bar{D}^2 D^2 \partial_\mu}{8\partial^4} - \bar{\theta}^a \theta^b \theta_b \frac{\bar{D}_a D^2}{2\partial^2} \quad (111)$$

and contains terms, proportional to $\bar{\theta}^a \theta^b \theta_b$.

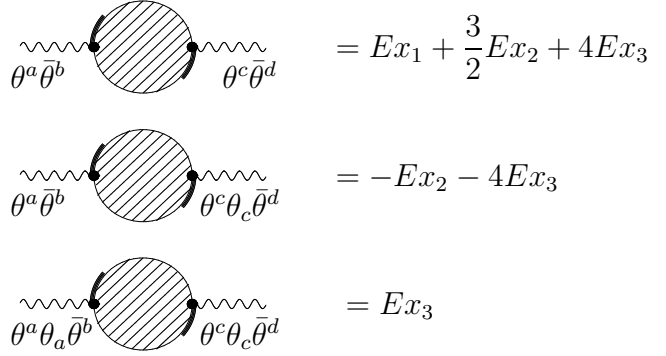
Also in the massive case it is necessary to take into account the effective diagrams


(112)

The first diagram corresponds to terms, proportional to $\bar{\theta}^a$ in Eq. (104). The second one contains a sum of subdiagrams with two adjacent external lines. These subdiagrams are presented in Appendix D.

Let us now write down the results for all diagrams (again, omitting $-\sum_I c_I$ for simplicity):

The diagrams contributing in the massless case are calculated similarly. Taking into account identities (106), the result can be written as



$$\left. \begin{aligned} &= Ex_1 + \frac{3}{2}Ex_2 + 4Ex_3 \\ &= -Ex_2 - 4Ex_3 \\ &= Ex_3 \end{aligned} \right\} = Ex_1 + \frac{1}{2}Ex_2 + Ex_3$$

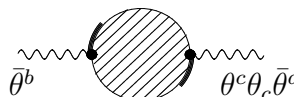
where

$$\begin{aligned} Ex_1 &= \frac{i}{2} \frac{d}{d \ln \Lambda} \text{Tr} \left\langle \theta^4 [y_\mu^*, I_0] \star [y_\mu^*, I_0] \star \right\rangle; \\ Ex_2 &= 2(\gamma^\mu)^c \frac{d}{d \ln \Lambda} \text{Tr} \left\langle \theta^4 \left([y_\mu^*, (I_1)_c] \star (\bar{I}_1)^d \star + [y_\mu^*, I_0] \star (I_1)_c \star (\bar{I}_1)^d \star \right) \right\rangle; \\ Ex_3 &= -2i \frac{d}{d \ln \Lambda} \text{Tr} \left\langle \theta^4 \left(-(\bar{I}_1)_d \star (I_2) \star (\bar{I}_1)^d \star - (\bar{I}_3)_d \star (\bar{I}_1)^d \star + 2(I_2)_d^c \star (I_1)_c \star (\bar{I}_1)^d \star \right. \right. \\ &\quad \left. \left. + 2(\bar{I}_1)_d \star (I_1)^c \star (I_1)_c \star (\bar{I}_1)^d \star \right) \right\rangle. \end{aligned} \quad (113)$$

Taking into account that

$$\star [\bar{\theta}_a, I_0] \star = [\bar{\theta}^a, \star]; \quad \text{Tr}(A[\bar{\theta}^a, B]) = \text{Tr}(\{A, \bar{\theta}^a\}B) \quad (114)$$

and using Eq. (106), we also obtain



$$= -i \frac{d}{d \ln \Lambda} \text{Tr} \left\langle \theta^4 \star \mathcal{V} \right\rangle \quad (115)$$

$$\left(\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ \frac{4}{\partial^2 + M^2} - \frac{i(\gamma^\mu)^{ab} D_a \bar{D}_b \partial_\mu}{(\partial^2 + M^2)^2} & 0 & 0 & 0 & \frac{MD^2}{(\partial^2 + M^2)^2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{MD^2}{(\partial^2 + M^2)^2} & \frac{4}{\partial^2 + M^2} - \frac{i(\gamma^\mu)^{ab} D_a \bar{D}_b \partial_\mu}{(\partial^2 + M^2)^2} & 0 & 0 \end{array} \right) \rangle.$$

The last diagram is calculated using expressions for the subdiagrams with two adjacent external lines, presented in Appendix D. The result is

$$\begin{aligned} \text{Diagram} &= -i \frac{d}{d \ln \Lambda} \text{Tr} \left\langle \theta^4 \star \frac{4M^2}{(\partial^2 + M^2)^2} I_0 \right\rangle \\ &+ i \frac{d}{d \ln \Lambda} \text{Tr} \left\langle \theta^4 \star \mathcal{V} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{MD^2}{(\partial^2 + M^2)^2} \\ 0 & 0 & 0 & 0 \\ 0 & \frac{MD^2}{(\partial^2 + M^2)^2} & 0 & 0 \end{pmatrix} \right\rangle. \end{aligned} \quad (116)$$

Now let us find a sum of these diagrams. Similar to the massless case

$$\begin{aligned} Ex_1 &= \frac{i}{2} \frac{d}{d \ln \Lambda} \text{Tr} \left\langle \theta^4 [y_\mu^*, [y_\mu^*, I_0] \star] - \theta^4 [y_\mu^*, [y_\mu^*, I_0]] \star \right\rangle = i \frac{d}{d \ln \Lambda} \text{Tr} \left\langle \theta^4 \frac{4M^2}{(\partial^2 + M^2)^2} I_0 \star \right\rangle \\ &- i \frac{d}{d \ln \Lambda} \text{Tr} \left\langle \theta^4 \mathcal{V} \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{i(\gamma^\mu)^{ab} D_a \bar{D}_b \partial_\mu}{(\partial^2 + M^2)^2} - \frac{4}{\partial^2 + M^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{i(\gamma^\mu)^{ab} D_a \bar{D}_b \partial_\mu}{(\partial^2 + M^2)^2} - \frac{4}{\partial^2 + M^2} & 0 \end{pmatrix} \star \right\rangle, \end{aligned} \quad (117)$$

where we take into account that

$$\begin{aligned} \theta^4 \left[y_\mu^*, \left[y_\mu^*, \frac{D^2 \bar{D}^2}{16(\partial^2 + M^2)} \right] \right] &= \theta^4 \left(-\frac{M^2 D^2 \bar{D}^2}{2(\partial^2 + M^2)^3} + \frac{i D^2 (\gamma^\mu)^{ab} \theta_a \bar{D}_b \partial_\mu}{(\partial^2 + M^2)^2} + \frac{2 D^2 \theta^a \theta_a}{\partial^2 + M^2} \right) \\ &= \theta^4 \left(-\frac{M^2 D^2 \bar{D}^2}{2(\partial^2 + M^2)^3} + \frac{2i(\gamma^\mu)^{ab} D_a \bar{D}_b \partial_\mu}{(\partial^2 + M^2)^2} - \frac{8}{\partial^2 + M^2} \right). \end{aligned} \quad (118)$$

Singularities, giving δ -functions, are certainly absent in the massive case.

Summing (115), (116), and (117) we obtain

$$\frac{i}{2} \frac{d}{d \ln \Lambda} \text{Tr} \left\langle \theta^4 [y_\mu^*, [y_\mu^*, I_0] \star] \right\rangle = 0. \quad (119)$$

Moreover, exactly as in the massless case for diagrams with n vertexes on the matter loop

$$\begin{aligned}
\frac{1}{2}Ex_2 + Ex_3 &= \frac{d}{d \ln \Lambda} \text{Tr} \left\langle \theta^4 \left[y_\mu^*, -\frac{1}{2}(\gamma^\mu)^{cd} (I_1)_c \star (\bar{I}_1)_d \star \right] \right\rangle_n - \frac{1}{n(n+1)(n+2)} \frac{d}{d \ln \Lambda} \\
&\times \text{Tr} \left\langle \theta^4 (\gamma^\mu)^{cd} \left[y_\mu^*, \star^3 [\bar{\theta}_d, \star] [\theta_c, I_0] + [\theta_c, \star^3] [\bar{\theta}_d, \star] I_0 + \star [\bar{\theta}_d, I_0] [\theta_c, \star^3] + [\theta_c, \star] [\bar{\theta}_d, I_0] \star^3 \right. \right. \\
&\left. \left. + I_0 [\bar{\theta}_d, \star^3] [\theta_c, \star] + [\theta_c, I_0] [\bar{\theta}_d, \star^3] \star + \star [\bar{\theta}_d, \star] [\theta_c, \star] + [\theta_c, \star] [\bar{\theta}_d, \star] \star \right] \right\rangle_n = 0. \quad (120)
\end{aligned}$$

(It is evident that there are no terms proportional to a δ -function in this case.) Taking into account a possibility of making cyclic permutations (72), we obtain that the contribution of the Pauli–Villars fields is given by the following integral of a total derivative (for diagrams with n vertexes on the matter loop to that external lines are attached):

$$\frac{d}{d \ln \Lambda} \text{Tr} \left\langle \theta^4 \left[y_\mu^*, \frac{i}{2} [y_\mu^*, I_0] \star - \sum_{a+b+2=n} \frac{(b+1)(\gamma^\mu)^{cd}}{n} (I_1)_c (\star)_a (\bar{I}_1)_d (\star)_b \right] \right\rangle_n = 0. \quad (121)$$

As earlier, the left hand side can be rewritten as

$$\frac{i}{2} \frac{d}{d \ln \Lambda} \text{Tr} \left\langle \theta^4 \left[y_\mu^*, \left[(y^\mu)^*, \ln(\star) \right] \right] \right\rangle = 0. \quad (122)$$

The final result for the sum of diagrams in that the external \mathbf{V} -lines are attached to a single loop of matter superfields is

$$i \frac{d}{d \ln \Lambda} \text{Tr} \left\langle \theta^4 \left[y_\mu^*, \left[(y^\mu)^*, \ln(\star) + \ln(\tilde{\star}) - \frac{1}{2} \sum_I c_I \ln(\star_I) \right] \right] \right\rangle - \text{terms with a } \delta\text{-function}, \quad (123)$$

where \star_I means that it is necessary to use the mass M_I in the definition of \star . The factorization of integrands into double total derivatives, which follows from this equation, agrees with the arguments presented in Ref. [22].

From Eq. (123) it is possible to explicitly construct an integral of a total derivative, which is equal to the sum of Feynman diagrams. In the three-loop approximation this is made in the following section.

9 Three-loop verification

In order to verify the expressions, obtained in the previous sections, we compare them with the explicit three-loop calculation, made in Ref. [21] by a different method. The result can be written as [29]

$$\frac{\beta(\alpha_0)}{\alpha_0^2} = \frac{d}{d \ln \Lambda} \left(d^{-1}(\alpha_0, \Lambda/p) - \alpha_0^{-1} \right) \Big|_{p=0} = 16\pi(A_1 + A_2 + A_3), \quad (124)$$

where⁸

⁸This result was also presented in Ref. [34], but some sign in A_3 were written incorrectly.

$$A_1 = -\frac{1}{2} \sum_I c_I \int \frac{d^4 q}{(2\pi)^4} \frac{1}{q^2} \frac{d}{dq^2} \frac{d}{d \ln \Lambda} \left(\ln(q^2 + M_I^2) + \frac{M_I^2}{q^2 + M_I^2} \right); \quad (125)$$

$$A_2 = -2e^2 \int \frac{d^4 q}{(2\pi)^4} \frac{1}{q^2} \frac{d}{dq^2} \frac{d}{d \ln \Lambda} \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 R_k^2} \left(\frac{1}{(k+q)^2} - \sum_J c_J \frac{q^4}{(q^2 + M_J^2)^2} \right. \\ \times \left. \frac{1}{((k+q)^2 + M_J^2)} \right) \left[R_k \left(1 + \frac{e^2}{4\pi^2} \ln \frac{\Lambda}{\mu} \right) - \int \frac{d^4 t}{(2\pi)^4} \frac{2e^2}{t^2 (k+t)^2} \right. \\ \left. + \sum_I c_I \int \frac{d^4 t}{(2\pi)^4} \frac{2e^2}{(t^2 + M_I^2)((k+t)^2 + M_I^2)} \right]; \quad (126)$$

$$A_3 = \int \frac{d^4 q}{(2\pi)^4} \frac{1}{q^2} \frac{d}{dq^2} \frac{d}{d \ln \Lambda} \int \frac{d^4 k}{(2\pi)^4} \frac{d^4 l}{(2\pi)^4} \frac{4e^4}{k^2 R_k l^2 R_l} \left\{ \frac{1}{(q+k)^2} \left[-\frac{1}{2(q+l)^2} \right. \right. \\ \left. + \frac{2k^2}{(q+k+l)^2(q+l)^2} - \frac{1}{(q+k+l)^2} \right] - \sum_I c_I \frac{q^4}{(q^2 + M_I^2)^2} \frac{1}{((q+k)^2 + M_I^2)} \times \\ \times \left[-\frac{1}{2((q+l)^2 + M_I^2)} + \frac{2k^2}{((q+k+l)^2 + M_I^2)((q+l)^2 + M_I^2)} \right. \\ \left. - \frac{1}{((q+k+l)^2 + M_I^2)} + \frac{2M_I^2}{((q+k)^2 + M_I^2)((q+k+l)^2 + M_I^2)} \right. \\ \left. + \frac{2M_I^2}{(q^2 + M_I^2)((q+l)^2 + M_I^2)} + \frac{2M_I^2}{((q+l)^2 + M_I^2)((q+k+l)^2 + M_I^2)} \right] \left. \right\}, \quad (127)$$

where $R_k \equiv R(k^2/\Lambda^2)$. Here A_1 is a one-loop result. A_2 is a sum of two-loop diagrams, three-loop diagrams with two loops of the matter superfields, and diagrams with insertions of counterterms arising from renormalization of the coupling constant. A_3 is a sum of three-loop diagrams with a single loop of the matter superfields.

The anomalous dimension can be written as

$$\gamma(\alpha_0) = -\frac{d \ln Z}{d \ln \Lambda} = \frac{d}{d \ln \Lambda} \left(\ln(ZG) - \ln Z \right) \Big|_{q=0} = \frac{d \ln G}{d \ln \Lambda} \Big|_{q=0}, \quad (128)$$

where Λ and the renormalized coupling constant α are considered as independent variables. The two-point Green function of the matter superfield in the two-loop approximation is given by the following integrals [21, 34]:

$$\ln G = - \int \frac{d^4 k}{(2\pi)^4} \frac{2e_0^2}{k^2 R_k (k+q)^2} \left[1 - \frac{1}{R_k} \int \frac{d^4 t}{(2\pi)^4} \frac{2e_0^2}{t^2 (k+t)^2} + \sum_I c_I \frac{1}{R_k} \int \frac{d^4 t}{(2\pi)^4} \right. \\ \times \left. \frac{2e_0^2}{(t^2 + M_I^2)((k+t)^2 + M_I^2)} \right] + \int \frac{d^4 k}{(2\pi)^4} \frac{d^4 l}{(2\pi)^4} \frac{e_0^4}{k^2 R_k l^2 R_l} \left(-\frac{2}{(q+k)^2 (q+l)^2} \right. \\ \left. - \frac{4}{(q+k)^2 (q+k+l)^2} + \frac{8k^2 - 4q^2}{(q+k)^2 (q+k+l)^2 (q+l)^2} \right). \quad (129)$$

Therefore, taking into account one-loop renormalization of the coupling constant, we obtain

$$\begin{aligned} \gamma(\alpha_0) = & -2e^2 \int \frac{d^4 k}{(2\pi)^4} \frac{d}{d \ln \Lambda} \frac{1}{k^4 R_k^2} \left[R_k \left(1 + \frac{e^2}{4\pi^2} \ln \frac{\Lambda}{\mu} \right) - \int \frac{d^4 t}{(2\pi)^4} \frac{2e^2}{t^2 (k+t)^2} + \right. \\ & \left. + \sum_I c_I \int \frac{d^4 t}{(2\pi)^4} \frac{2e^2}{(t^2 + M_I^2)((k+t)^2 + M_I^2)} \right] - \int \frac{d^4 k}{(2\pi)^4} \frac{d^4 l}{(2\pi)^4} \frac{d}{d \ln \Lambda} \frac{4e^4 k_\mu l_\mu}{k^4 R_k l^4 R_l (k+l)^2}. \end{aligned} \quad (130)$$

This expression is finite both in the UV and IR regions. The UV finiteness is ensured by the regularization. The integral is IR finite due to the differentiation with respect to $\ln \Lambda$, which should be performed before the integration.

Taking the integrals of total derivatives in Eqs. (125) — (127) using the identity

$$\int \frac{d^4 q}{(2\pi)^4} \frac{1}{q^2} \frac{d}{dq^2} f(q^2) = \frac{1}{16\pi^2} \left(f(q^2 = \infty) - f(q^2 = 0) \right), \quad (131)$$

we obtain

$$\frac{\beta(\alpha_0)}{\alpha_0^2} = \frac{1}{\pi} \left(1 - \frac{d \ln G}{d \ln \Lambda} \Big|_{q=0} \right) + O(\alpha_0^3). \quad (132)$$

As a consequence,

$$\beta(\alpha) = \frac{\alpha^2}{\pi} \left(1 - \gamma(\alpha) \right) + O(\alpha^5). \quad (133)$$

In order to verify Eq. (130), it was compared with the result of the calculation made with the dimensional reduction. (Such calculation can be made using Eq. (129).) Using the standard technique of the dimensional reduction one obtain

$$\gamma_{\text{DRED}}(\alpha) = -\frac{\alpha}{\pi} + \frac{\alpha^2}{\pi^2} + O(\alpha^3). \quad (134)$$

This result up to notations⁹ agrees with the calculation made in [5].

From the other side, it is possible to calculate the two-point Green function of the gauge superfield using Eqs. (89) and (121). Taking into account that a contribution of diagrams with a $\tilde{\phi}$ -loop is equal to a contribution of diagrams with a ϕ -loop, we obtain

$$\begin{aligned} & 4i \frac{d}{d \ln \Lambda} \text{Tr} \left\langle \theta^4 \left[y_\mu^*, (e^{2V} - 1) \frac{\bar{D}^2 D^2 \partial^\mu}{16 \partial^4} * \right] \right\rangle + \text{terms with a } \delta\text{-function} \\ & = -32\pi \int \frac{d^4 q}{(2\pi)^4} \frac{1}{q^2} \frac{d}{dq^2} \int \frac{d^4 k}{(2\pi)^4} \frac{d}{d \ln \Lambda} \frac{e^2}{k^2 R_k^2 (k+q)^2} \left[R_k \left(1 + \frac{e^2}{4\pi^2} \ln \frac{\Lambda}{\mu} \right) \right. \\ & \quad \left. - 2e^2 \left(\int \frac{d^4 t}{(2\pi)^4} \frac{1}{t^2 (k+t)^2} - \sum_I c_I \int \frac{d^4 t}{(2\pi)^4} \frac{1}{(t^2 + M_I^2)((k+t)^2 + M_I^2)} \right) \right] \\ & \quad + 64\pi \int \frac{d^4 q}{(2\pi)^4} \frac{1}{q^2} \frac{d}{dq^2} \int \frac{d^4 k}{(2\pi)^4} \frac{d^4 l}{(2\pi)^4} \frac{d}{d \ln \Lambda} \frac{e^4}{k^2 R_k l^2 R_l} \left(\frac{1}{(q+k)^2 (q+k+l)^2} \right. \\ & \quad \left. - \frac{(2q+k+l)^2}{(q+k)^2 (q+l)^2 (q+k+l)^2} \right). \end{aligned} \quad (135)$$

⁹In order to obtain the results of Ref. [5] it is necessary to set $\alpha = g^2/4\pi$, $\gamma(\alpha) = 2\gamma(g)$, $\beta(\alpha) = g\beta(g)/2\pi$.

The last term in Eq. (89) is

$$\begin{aligned}
& -4 \sum_{a+b+2=n} \frac{d}{d \ln \Lambda} \frac{b+1}{n} (\gamma^\mu)^{cd} \text{Tr} \left\langle \theta^4 \left[y_\mu^*, (I_1)_c (*)_a (\bar{I}_1)_d (*)_b \right] \right\rangle \\
& \quad + \text{terms with a } \delta\text{-function} \\
& = 32\pi \int \frac{d^4 q}{(2\pi)^4} \frac{1}{q^2} \frac{d}{dq^2} \int \frac{d^4 k}{(2\pi)^4} \frac{d^4 l}{(2\pi)^4} \frac{d}{d \ln \Lambda} \frac{e^4}{k^2 R_k l^2 R_l} \frac{1}{(q+k)^2 (q+l)^2} \\
& + 16\pi \int \frac{d^4 q}{(2\pi)^4} \frac{d^4 k}{(2\pi)^4} \frac{d^4 l}{(2\pi)^4} \frac{d}{d \ln \Lambda} \frac{e^4}{k^2 R_k l^2 R_l} \frac{\partial}{\partial q^\mu} \frac{(2q+k+l)^\mu}{q^2 (q+k)^2 (q+l)^2 (q+k+l)^2} \quad (136)
\end{aligned}$$

Contributions of diagrams with a Pauli–Villars loop can be calculated similarly:

$$\begin{aligned}
& -\frac{i}{2} \sum_J c_J \frac{d}{d \ln \Lambda} \text{Tr} \left\langle \theta^4 \left[y_\mu^*, [y_\mu^*, I_0] \star \right] \right\rangle = \\
& = 32\pi \sum_J c_J \int \frac{d^4 q}{(2\pi)^4} \frac{1}{q^2} \frac{d}{dq^2} \int \frac{d^4 k}{(2\pi)^4} \frac{d}{d \ln \Lambda} \frac{e^2}{k^2 R_k^2} \frac{q^4}{(q^2 + M_J^2)((k+q)^2 + M_J^2)} \left[R_k \left(1 \right. \right. \\
& \left. \left. + \frac{e^2}{4\pi^2} \ln \frac{\Lambda}{\mu} \right) - \int \frac{d^4 t}{(2\pi)^4} \frac{2e^2}{t^2 (k+t)^2} + \sum_I c_I \int \frac{d^4 t}{(2\pi)^4} \frac{2e^2}{(t^2 + M_I^2)((k+t)^2 + M_I^2)} \right] \\
& - 64\pi \sum_I c_I \int \frac{d^4 q}{(2\pi)^4} \frac{1}{q^2} \frac{d}{dq^2} \int \frac{d^4 k}{(2\pi)^4} \frac{d^4 l}{(2\pi)^4} \frac{d}{d \ln \Lambda} \frac{e^4}{k^2 R_k l^2 R_l} \frac{q^4}{(q^2 + M_I^2)^2} \\
& \times \left(\frac{1}{((q+k+l)^2 + M_I^2)((q+k)^2 + M_I^2)} - \frac{(2q+k+l)^2 + 2M_I^2}{((q+k)^2 + M_I^2)((q+l)^2 + M_I^2)} \right. \\
& \left. \times \frac{1}{((q+k+l)^2 + M_I^2)} + \frac{2M_I^2}{(q^2 + M_I^2)((q+k)^2 + M_I^2)((q+l)^2 + M_I^2)} \right. \\
& \left. + \frac{2M_I^2}{((q+k)^2 + M_I^2)^2((q+k+l)^2 + M_I^2)} \right) \\
& - 32\pi \sum_I c_I \int \frac{d^4 q}{(2\pi)^4} \frac{1}{q^2} \frac{d}{dq^2} \int \frac{d^4 k}{(2\pi)^4} \frac{d^4 l}{(2\pi)^4} \frac{d}{d \ln \Lambda} \frac{e^4}{k^2 R_k l^2 R_l} \frac{q^4}{(q^2 + M^2)^2((q+k)^2 + M_I^2)} \\
& \times \frac{1}{((q+l)^2 + M_I^2)} - 16\pi \sum_I c_I \int \frac{d^4 q}{(2\pi)^4} \frac{d^4 k}{(2\pi)^4} \frac{d^4 l}{(2\pi)^4} \frac{d}{d \ln \Lambda} \frac{e^4}{k^2 R_k l^2 R_l} \frac{\partial}{\partial q^\mu} \frac{1}{(q^2 + M_I^2)} \\
& \times \frac{(2q+k+l)^\mu}{((q+k)^2 + M_I^2)((q+l)^2 + M_I^2)((q+k+l)^2 + M_I^2)}; \quad (137)
\end{aligned}$$

$$\sum_I c_I \sum_{a+b+2=n} \frac{d}{d \ln \Lambda} \frac{b+1}{n} (\gamma^\mu)^{cd} \text{Tr} \left\langle \theta^4 \left[y_\mu^*, (I_1)_c (*)_a (\bar{I}_1)_d (*)_b \right] \right\rangle = 0. \quad (138)$$

Summing all these contributions with the one-loop result, we obtain

$$\frac{\beta(\alpha_0)}{\alpha_0^2} = 16\pi(A_1 + A_2 + A_3). \quad (139)$$

Thus, the general results, presented above, agree with the explicit three-loop calculations, made by a different method.

Also it is possible to verify expression (123). After a calculation of Feynman diagrams we obtained

$$\begin{aligned}
\text{tr} \langle \ln(\star) \rangle = & \left\{ - \int \frac{d^4 k}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} \frac{e^2}{k^2 R_k^2 (q^2 + M^2) ((q+k)^2 + M^2)} \left[R_k \left(1 + \frac{e^2}{4\pi^2} \ln \frac{\Lambda}{\mu} \right) \right. \right. \\
& - 2e^2 \left(\int \frac{d^4 t}{(2\pi)^4} \frac{1}{t^2 (k+t)^2} - \sum_J c_J \int \frac{d^4 t}{(2\pi)^4} \frac{1}{(t^2 + M_J^2) ((k+t)^2 + M_J^2)} \right) \Big] \\
& + \int \frac{d^4 q}{(2\pi)^4} \frac{d^4 k}{(2\pi)^4} \frac{d^4 l}{(2\pi)^4} \frac{e^4}{k^2 R_k l^2 R_l} \left(- \frac{q^2 - M^2}{(q^2 + M^2)^2 ((q+k)^2 + M^2) ((q+l)^2 + M^2)} \right. \\
& - \frac{(q+k)^2 - M^2}{(q^2 + M^2) ((q+k)^2 + M^2)^2 ((q+l+k)^2 + M^2)} + \frac{4}{3(q^2 + M^2) ((q+k)^2 + M^2)} \\
& \times \frac{1}{((q+l)^2 + M^2)} + \frac{8}{3(q^2 + M^2) ((q+k)^2 + M^2) ((q+k+l)^2 + M^2)} \\
& \left. \left. - \frac{(2q+k+l)^2 + 2M^2}{(q^2 + M^2) ((q+k)^2 + M^2) ((q+l)^2 + M^2) ((q+k+l)^2 + M^2)} \right) \right\} \\
& \times \frac{i(\bar{D}^2 D^2 + D^2 \bar{D}^2)}{8} e^{-q_\alpha (x-y)^\alpha} \delta^4(\theta_x - \theta_y). \tag{140}
\end{aligned}$$

where tr is a usual matrix trace, which (unlike Tr) does not contain $\int d^8 x$. As a consequence, after some simple algebra we obtain

$$\begin{aligned}
\frac{\beta(\alpha_0)}{\alpha_0^2} = & 2\pi \frac{d}{d \ln \Lambda} \sum_I c_I \int \frac{d^4 q}{(2\pi)^4} \frac{\partial}{\partial q^\mu} \frac{\partial}{\partial q_\mu} \frac{\ln(q^2 + M^2)}{q^2} + 4\pi \frac{d}{d \ln \Lambda} \int \frac{d^4 q}{(2\pi)^4} \frac{d^4 k}{(2\pi)^4} \frac{e^2}{k^2 R_k^2} \\
& \times \frac{\partial}{\partial q^\mu} \frac{\partial}{\partial q_\mu} \left(\frac{1}{q^2 (k+q)^2} - \sum_I c_I \frac{1}{(q^2 + M_I^2) ((k+q)^2 + M_I^2)} \right) \left[R_k \left(1 + \frac{e^2}{4\pi^2} \ln \frac{\Lambda}{\mu} \right) \right. \\
& - 2e^2 \left(\int \frac{d^4 t}{(2\pi)^4} \frac{1}{t^2 (k+t)^2} - \sum_J c_J \int \frac{d^4 t}{(2\pi)^4} \frac{1}{(t^2 + M_J^2) ((k+t)^2 + M_J^2)} \right) \Big] \\
& + 4\pi \frac{d}{d \ln \Lambda} \int \frac{d^4 q}{(2\pi)^4} \frac{d^4 k}{(2\pi)^4} \frac{d^4 l}{(2\pi)^4} \frac{e^4}{k^2 R_k l^2 R_l} \frac{\partial}{\partial q^\mu} \frac{\partial}{\partial q_\mu} \left\{ \left(- \frac{2k^2}{q^2 (q+k)^2 (q+l)^2 (q+k+l)^2} \right. \right. \\
& + \frac{2}{q^2 (q+k)^2 (q+l)^2} \Big) - \sum_I c_I \left(- \frac{2(k^2 + M_I^2)}{(q^2 + M_I^2) ((q+k)^2 + M_I^2) ((q+l)^2 + M_I^2)} \right. \\
& + \frac{1}{((q+k+l)^2 + M_I^2)} + \frac{2}{(q^2 + M_I^2) ((q+k)^2 + M_I^2) ((q+l)^2 + M_I^2)} - \frac{1}{(q^2 + M_I^2)^2} \\
& \left. \left. \times \frac{4M_I^2}{((q+k)^2 + M_I^2) ((q+l)^2 + M_I^2)} \right) \right\} - \text{integrals of } \delta\text{-singularities.} \tag{141}
\end{aligned}$$

Then, using the equation

$$\int \frac{d^4 q}{(2\pi)^4} \left\{ \frac{\partial}{\partial q_\mu} \left(\frac{q_\mu}{q^4} f(q) \right) - 2\pi^2 \delta^4(q) f(q) \right\} = 2 \int \frac{d^4 q}{(2\pi)^4} \frac{1}{q^2} \frac{df}{dq^2}, \tag{142}$$

we again obtain

$$\frac{\beta(\alpha_0)}{\alpha_0^2} = 16\pi(A_1 + A_2 + A_3). \quad (143)$$

10 Conclusion

In this paper we have proved that the β -function in $N = 1$ supersymmetric electrodynamics, regularized by higher derivatives, is given by integrals of total derivatives in each order of the perturbation theory. In particular, explicit expressions for all total derivatives are obtained. Having compared them with the explicit three-loop calculation (made by a different method) we obtained the complete agreement.

Factorization of integrands into total derivatives is the origin of the exact NSVZ β -function, because one of the loop integrals can be taken. After this the β -function in n -th loop is related with the anomalous dimension in $n - 1$ -th loop. This was also proved in this paper by explicit summation of Feynman diagrams.

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A Derivation of expression for the two-point function

Splitting the classical action S into a quadratic part and an interaction S_I , it is possible to present the generating functional Z in the form

$$Z = \exp \left\{ iS_I \left(\frac{1}{i} \frac{\delta}{\delta J}, \frac{1}{i} \frac{\delta}{\delta j} \right) \right\} \exp \left\{ i \int d^8x \left(-J \frac{e_0^2}{\partial^2 R} J + j^* \frac{1}{\partial^2} j + \tilde{j}^* \frac{1}{\partial^2} \tilde{j} \right) \right\} \equiv e^{iS_I} Z_0, \quad (144)$$

where the Pauli–Villars determinants are omitted for simplicity. Differentiating this generating functional with respect to the source J we obtain

$$\begin{aligned} \frac{\delta Z}{\delta J_x} &= \exp(iS_I) (-2ie_0^2) \frac{1}{\partial_x^2 R} J_x Z_0 = -2ie_0^2 \frac{1}{\partial_x^2 R} J_x Z - 2ie_0^2 \frac{1}{\partial_x^2 R} \frac{\delta S}{\delta V_x} \left(\frac{1}{i} \frac{\delta}{\delta J}, \frac{1}{i} \frac{\delta}{\delta j} \right) Z; \\ \frac{\delta^2 Z}{\delta J_x \delta J_y} &= \exp(iS_I) \left(-\frac{2ie_0^2}{\partial^2 R} \delta_{xy}^8 - 4e_0^2 \frac{1}{\partial_x^2 R} J_x \frac{1}{\partial_y^2 R} J_y \right) Z_0. \end{aligned} \quad (145)$$

As a consequence

$$\begin{aligned} \frac{1}{Z} \frac{\delta Z}{\delta J_x} &= -\frac{2ie_0^2}{\partial_x^2 R} J_x - \frac{2ie_0^2}{\partial_x R} \left\langle \frac{\delta S_I}{\delta V_x} \right\rangle; \\ \frac{1}{Z} \frac{\delta^2 Z}{\delta J_x \delta J_y} &= -\frac{2ie_0^2}{\partial_x^2 R} \delta_{xy}^8 - 4e_0^4 \left\langle \frac{1}{\partial_x^2 R} \frac{\delta S_I}{\delta V_x} \frac{1}{\partial_y^2 R} \frac{\delta S_I}{\delta V_y} \right\rangle + 4e_0^2 i \frac{1}{\partial_x^2 R} \frac{1}{\partial_y^2 R} \left\langle \frac{\delta^2 S_I}{\delta V_x \delta V_y} \right\rangle. \end{aligned} \quad (146)$$

(In the last equation we set $J = 0$.) Therefore, if we denote an argument of the effective action by \mathbf{V} , the two-point Green function of the gauge superfield will satisfy

$$\begin{aligned} \left(\frac{\delta^2 \Gamma}{\delta \mathbf{V}_x \delta \mathbf{V}_y} \right)^{-1} &= -\frac{\delta^2 W}{\delta J_x \delta J_y} = i \frac{\delta^2 \ln Z}{\delta J_x \delta J_y} = \\ &= \frac{2e_0^2}{\partial_x^2 R} \delta_{xy}^8 - \left(4ie_0^4 \left\langle \frac{1}{\partial_x^2 R} \frac{\delta S_I}{\delta V_x} \frac{1}{\partial_y^2 R} \frac{\delta S_I}{\delta V_y} \right\rangle + 4e_0^4 \frac{1}{\partial_x^2 R} \frac{1}{\partial_y^2 R} \left\langle \frac{\delta^2 S_I}{\delta V_x \delta V_y} \right\rangle \right)_{\text{connected}}. \end{aligned} \quad (147)$$

The inverse matrix is evidently given by

$$\frac{\delta^2 \Gamma}{\delta \mathbf{V}_x \delta \mathbf{V}_y} = \frac{1}{2e_0^2} \partial^2 R \delta_{xy}^8 + \left\langle i \frac{\delta S_I}{\delta V_x} \frac{\delta S_I}{\delta V_y} + \frac{\delta^2 S_I}{\delta V_x \delta V_y} \right\rangle_{\text{1PI}}, \quad (148)$$

where the symbol 1PI means that it is necessary to keep only one-particle irreducible graphs in this expression. Thus, a part of the effective action quadratic in the gauge superfield can be written as

$$\Gamma_{\mathbf{V}}^{(2)} = \frac{1}{4e^2} \int d^8 x \mathbf{V} \partial^2 R \mathbf{V} + \frac{1}{2} \int d^8 x d^8 y \mathbf{V}_x \mathbf{V}_y \left\langle i \frac{\delta S_I}{\delta V_x} \frac{\delta S_I}{\delta V_y} + \frac{\delta^2 S_I}{\delta V_x \delta V_y} \right\rangle_{\text{1PI}}. \quad (149)$$

The Pauli–Villars determinants can be considered similarly. In this case S_I contains the Pauli–Villars fields, and $\sum_J c_J$ should be also included.

B Prove of identity (62)

Let us consider

$$\begin{aligned} X \equiv & \text{Tr} \left\{ \theta^4 \left((\gamma^\mu)^{ab} [y_\mu^*, A] [\bar{\theta}_b, B] [\theta_a, C] + (\gamma^\mu)^{ab} (-1)^{PA} [\theta_a, B] [\bar{\theta}_b, C] [y_\mu^*, A] \right. \right. \\ & \left. \left. - 4i [\theta^a, [\theta_a, A]] [\bar{\theta}^b, B] [\bar{\theta}_b, C] \right) \right\} + \text{cyclic perm. of } A, B, C, \end{aligned} \quad (150)$$

where A , B , and C are differential operators, containing supersymmetric covariant derivatives. It is important that they do not explicitly depend on θ . Certainly, we assume that

$$P_A + P_B + P_C = 0 \pmod{2}. \quad (151)$$

Using the identities

$$\begin{aligned} \text{Tr}([y_\mu^*, A] B) &= -\text{Tr}(A[y_\mu^*, B]); \\ (-1)^{PB} [[A, B], C] &+ (-1)^{PA} [[C, A], B] + (-1)^{PC} [[B, C], A] = 0, \end{aligned} \quad (152)$$

we obtain

$$\begin{aligned}
X = & (\gamma^\mu)^{ab} \text{Tr} \left\{ \theta^4 \left(\left[y_\mu^*, A[\bar{\theta}_b, B][\theta_a, C] + (-1)^{P_A} [\theta_a, B][\bar{\theta}_b, C] A \right] - A[\bar{\theta}_b, [y_\mu^*, B]] \right. \right. \\
& \times [\theta_a, C] - A[\bar{\theta}_b, B][\theta_a, [y_\mu^*, C]] - (-1)^{P_A} [\theta_a, [y_\mu^*, B]][\bar{\theta}_b, C] A - (-1)^{P_A} [\theta_a, B] \\
& \left. \left. [\bar{\theta}_b, [y_\mu^*, C]] A \right) \right\} - 4i \text{Tr} \left(\theta^4 [\theta^a, [\theta_a, A]] [\bar{\theta}^b, B][\bar{\theta}_b, C] \right) + \text{cyclic perm. of } A, B, C.
\end{aligned} \tag{153}$$

The similar operation is repeated for θ -s in double commutators:

$$\begin{aligned}
X = & (\gamma^\mu)^{ab} \text{Tr} \left\{ \theta^4 \left(\left[y_\mu^*, A[\bar{\theta}_b, B][\theta_a, C] + (-1)^{P_A} [\theta_a, B][\bar{\theta}_b, C] A \right] + (-1)^{P_A} [\bar{\theta}_b, A] \right. \right. \\
& \times [y_\mu^*, B][\theta_a, C] + (-1)^{P_B} A[y_\mu^*, B][\bar{\theta}_b, [\theta_a, C]] - (-1)^{P_C} [\theta_a, A][\bar{\theta}_b, B][y_\mu^*, C] \\
& - (-1)^{P_B} A[\theta_a, [\bar{\theta}_b, B]][y_\mu^*, C] - [y_\mu^*, B][\bar{\theta}_b, C][\theta_a, A] + (-1)^{P_C} [y_\mu^*, B][\theta_a, [\bar{\theta}_b, C]] A \\
& \left. \left. - (-1)^{P_C} [\bar{\theta}_b, [\theta_a, B]][y_\mu^*, C] A + (-1)^{P_B} [\theta_a, B][y_\mu^*, C][\bar{\theta}_b, A] \right) \right\} \\
& - 4i \text{Tr} \left(\theta^4 [\theta^a, [\theta_a, A]] [\bar{\theta}^b, B][\bar{\theta}_b, C] \right) + \text{cyclic perm. of } A, B, C.
\end{aligned} \tag{154}$$

In addition to θ^4 , the commutators with y_μ^* give one more degree of θ :

$$[y_\mu^*, A] = -2i(\gamma^\mu)^{ab} \theta_a [\bar{\theta}_b, A] + O(\theta^0). \tag{155}$$

That is why it is necessary to be careful commuting θ^4 with A , B and C . For example, taking into account that all expressions containing θ in less than fourth power vanish, we obtain

$$\begin{aligned}
(-1)^{P_A} (\gamma^\mu)^{ab} \text{Tr} \theta^4 [\bar{\theta}_b, A][y_\mu^*, B][\theta_a, C] &= (-1)^{P_B+1} (\gamma^\mu)^{ab} \text{Tr} \left(\theta^4 [\theta_a, C][\bar{\theta}_b, A][y_\mu^*, B] \right. \\
& \left. - 2\bar{\theta}^c \bar{\theta}_c \theta^d [\theta_a, [\theta_a, C]][\bar{\theta}_b, A] (-2i)(\gamma_\mu)^{ef} \theta_e [\bar{\theta}_f, B] \right) \\
&= \text{Tr} \left((-1)^{P_B+1} \theta^4 (\gamma^\mu)^{ab} [\theta_a, C][\bar{\theta}_b, A][y_\mu^*, B] + 4i\theta^4 [\theta^a, [\theta_a, C]][\bar{\theta}^b, A][\bar{\theta}_b, B] \right).
\end{aligned} \tag{156}$$

Similarly one can derive the following identities:

$$(-1)^{P_B} (\gamma^\mu)^{ab} \text{Tr} \left(\theta^4 [\theta_a, B][y_\mu^*, C][\bar{\theta}_b, A] \right) \tag{157}$$

$$\begin{aligned}
&= \text{Tr} \left(-\theta^4 (\gamma^\mu)^{ab} [y_\mu^*, C][\bar{\theta}_b, A][\theta_a, B] + 4i\theta^4 [\theta^a, [\theta_a, B]][\bar{\theta}^b, C][\bar{\theta}_b, A] \right); \\
&(-1)^{P_B+1} (\gamma^\mu)^{ab} \text{Tr} \left(\theta^4 A[\theta_a, [\bar{\theta}_b, B]][y_\mu^*, C] \right)
\end{aligned} \tag{158}$$

$$\begin{aligned}
&= \text{Tr} \left((-1)^{P_C+1} \theta^4 (\gamma^\mu)^{ab} [\theta_a, [\bar{\theta}_b, B]][y_\mu^*, C] A - 4i(-1)^{P_A} \theta^4 [\theta^a, A][\theta_a, [\bar{\theta}^b, B]][\bar{\theta}_b, C] \right); \\
&(-1)^{P_B} (\gamma^\mu)^{ab} \text{Tr} \left(\theta^4 A[y_\mu^*, B][\bar{\theta}_b, [\theta_a, C]] \right)
\end{aligned} \tag{159}$$

$$= (-1)^{P_C} \text{Tr} \left(\theta^4 (\gamma^\mu)^{ab} [y_\mu^*, B][\bar{\theta}_b, [\theta_a, C]] A + 4i\theta^4 [\theta^a, A][\bar{\theta}_b, B][\bar{\theta}^b, [\theta_a, C]] \right).$$

Using these equations, after some algebraic transformations X can be rewritten as

$$\begin{aligned}
X = & \text{Tr} \left\{ \theta^4 (\gamma^\mu)^{ab} \left([y_\mu^*, A[\bar{\theta}_b, B]\{\theta_a, C\} + (-1)^{P_A} [\theta_a, B]\{\bar{\theta}_b, C\}A \right] - 2[y_\mu^*, A][\bar{\theta}_b, B] \right. \\
& \times [\theta_a, C] - 2(-1)^{P_A} [\theta_a, B]\{\bar{\theta}_b, C\}[y_\mu^*, A] \Big) + 8i\theta^4 [\theta^a, [\theta_a, A]]\{\bar{\theta}^b, B\}[\bar{\theta}_b, C] \Big\} \\
& + \text{cyclic perm. of } A, B, C.
\end{aligned} \tag{160}$$

Comparing this expression with the definition of X , we obtain

$$\begin{aligned}
X = & -2X + \left(\text{Tr} \left(\theta^4 (\gamma_\mu)^{ab} [y_\mu^*, A[\bar{\theta}_b, B]\{\theta_a, C\} + (-1)^{P_A} [\theta_a, B]\{\bar{\theta}_b, C\}A \right] \right. \\
& \left. + \text{cyclic perm. of } A, B, C \right).
\end{aligned} \tag{161}$$

Therefore,

$$\begin{aligned}
X = & \frac{1}{3} \text{Tr} \left(\theta^4 (\gamma_\mu)^{ab} [y_\mu^*, A[\bar{\theta}_b, B]\{\theta_a, C\} + (-1)^{P_A} [\theta_a, B]\{\bar{\theta}_b, C\}A \right] \\
& + \text{cyclic perm. of } A, B, C.
\end{aligned} \tag{162}$$

This completes the proof.

C Simplification of expression (67)

Let us calculate

$$\begin{aligned}
& -\frac{2(\gamma^\mu)^{cd}}{n(n+1)(n+2)} \text{Tr} \left\langle \theta^4 \left[y_\mu^*, {}^4(\bar{I}_1)_d * (I_1)_c + \left({}^3(I_1)_c * + {}^2(I_1)_c {}^2 + * (I_1)_c {}^3 \right) * \right. \right. \\
& \times (\bar{I}_1)_d * I_0 + * (\bar{I}_1)_d \left({}^3(I_1)_c * + {}^2(I_1)_c {}^2 + * (I_1)_c {}^3 \right) + * (I_1)_c * (\bar{I}_1)_d {}^3 \\
& + I_0 \left({}^3(\bar{I}_1)_d * + {}^2(\bar{I}_1)_d {}^2 + * (\bar{I}_1)_d {}^3 \right) * (I_1)_c * + (I_1)_c \left({}^3(\bar{I}_1)_d * + {}^2(\bar{I}_1)_d {}^2 \right. \\
& \left. \left. + * (\bar{I}_1)_d {}^3 \right) * + {}^2(\bar{I}_1)_d {}^2 (I_1)_c * + * (I_1)_c {}^2 (\bar{I}_1)_d {}^2 \right] \Big\rangle_n
\end{aligned} \tag{163}$$

for a diagram containing n vertexes on the matter loop to that external lines are attached. Due to a possibility of making cyclic permutations (72)

$$(\gamma^\mu)^{cd} \text{Tr} \left\langle \left[y_\mu^*, (I_1)_c (*)_a (\bar{I}_1)_d (*)_b \right] \right\rangle_n = -(\gamma^\mu)^{cd} \text{Tr} \left\langle \left[y_\mu^*, \left\langle (\bar{I}_1)_d (*)_b (I_1)_c (*)_a \right\rangle \right] \right\rangle_n. \tag{164}$$

Therefore, using the identity

$$* I_0 * = - * + {}^2 \tag{165}$$

and Eq. (28), the considered expression can be written as

$$\sum_{a+b+2=n} c_a(\gamma^\mu)^{cd} \text{Tr} \left\langle \left[y_\mu^*, (I_1)_c(*)_a (\bar{I}_1)_d(*)_b \right] \right\rangle_n. \quad (166)$$

(Two vertexes correspond to $(I_1)_c$ and $(\bar{I}_1)_d$.) In order to find the coefficients c_a , it is necessary to calculate all $(*)_a$ and $(*)_b$, using Eq. (28). The result is proportional to

$$\begin{aligned} & -\frac{1}{6}(a+1)(a+2)(a+3) - \frac{1}{2}(b+1)(b+2)(a+1) - \frac{1}{2}(b+1)(a+1)(a+2) - \frac{1}{6}(a+1) \\ & \times (a+2)(a+3) + \frac{1}{6}(b+1)(b+2)(b+3)(a+1) + \frac{1}{4}(b+1)(b+2)(a+1)(a+2) \\ & + \frac{1}{6}(b+1)(a+1)(a+2)(a+3) - \frac{1}{2}(a+1)(b+1)(b+2) - \frac{1}{2}(a+1)(a+2)(b+1) \\ & - \frac{1}{6}(a+1)(a+2)(a+3) + \frac{1}{6}(b+1)(b+2)(b+3) + \frac{1}{2}(a+1)(a+2)(b+1) \\ & + \frac{1}{2}(a+1)(b+1)(b+2) + \frac{1}{6}(b+1)(b+2)(b+3) - \frac{1}{6}(b+1)(a+1)(a+2)(a+3) \\ & - \frac{1}{4}(a+1)(a+2)(b+1)(b+2) - \frac{1}{6}(a+1)(b+1)(b+2)(b+3) + \frac{1}{2}(a+1)(a+2)(b+1) \\ & + \frac{1}{2}(a+1)(b+1)(b+2) + \frac{1}{6}(b+1)(b+2)(b+3) - \frac{1}{2}(a+1)(a+2)(b+1) \\ & + \frac{1}{2}(a+1)(b+1)(b+2) \\ & = -\frac{1}{2}(a+1)(a+2)(a+3) - \frac{1}{2}(a+1)(a+2)(b+1) + \frac{1}{2}(a+1)(b+1)(b+2) \\ & + \frac{1}{2}(b+1)(b+2)(b+3) = \frac{1}{2}(b-a)(n+1)(n+2), \end{aligned} \quad (167)$$

where the sequence of terms in the first expression corresponds to the one in Eq. (163). Therefore, expression (163) can be rewritten as

$$- \sum_{a+b+2=n} \frac{b-a}{n} (\gamma^\mu)^{cd} \text{Tr} \left\langle \left[y_\mu^*, (I_1)_c(*)_a (\bar{I}_1)_d(*)_b \right] \right\rangle. \quad (168)$$

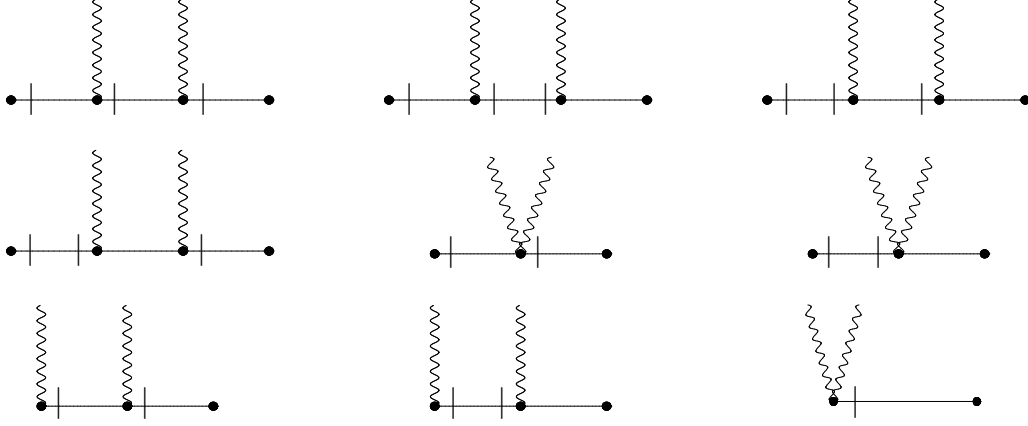
Adding to this expression the first term in Eq. (67), we obtain that the total derivative in Eq. (67) can be rewritten in the following form:

$$- \sum_{a+b+2=n} \frac{2(b+1)}{n} (\gamma^\mu)^{cd} \text{Tr} \left\langle \left[y_\mu^*, (I_1)_c(*)_a (\bar{I}_1)_d(*)_b \right] \right\rangle. \quad (169)$$

D Summation of subdiagrams with two external lines

Here we describe, how subdiagrams with two external lines are split into groups, convenient for the calculation, and present results of this calculation. (Expressions for the left vertexes are omitted for simplicity.)

1. Subdiagrams with a chiral left end and an antichiral right end:

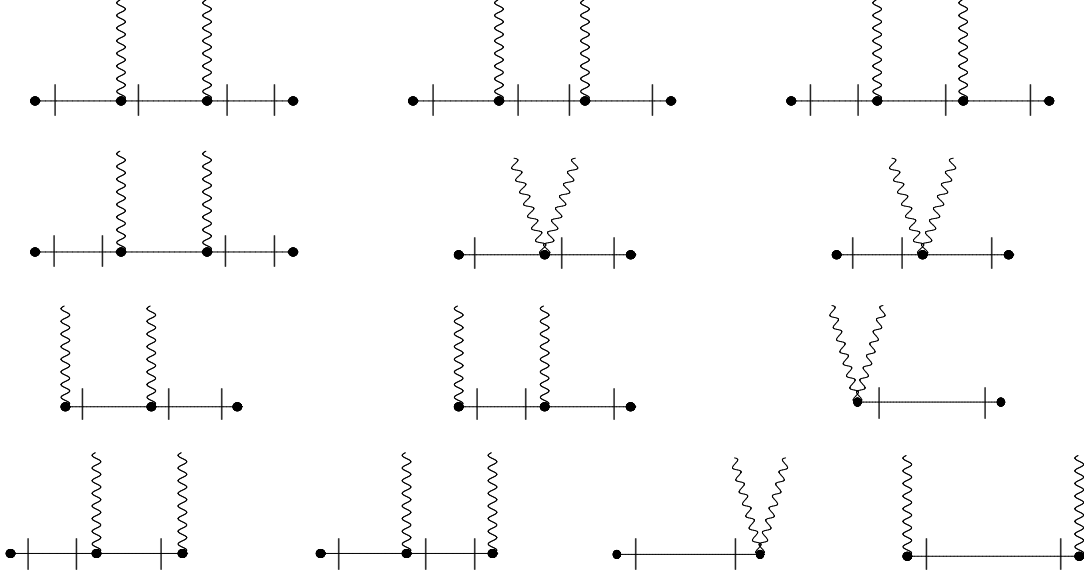


After the substitution $\mathbf{V} \rightarrow \theta^4$ the sum of these diagrams is written as

$$\theta^4 \frac{M^2 \bar{D}^2 D^2}{8(\partial^2 + M^2)^3} = \theta^4 \frac{2M^2}{(\partial^2 + M^2)^2} \cdot \frac{\bar{D}^2 D^2}{16(\partial^2 + M^2)}. \quad (170)$$

In particular, this means that in the massless case a sum of such diagrams is equal to 0.

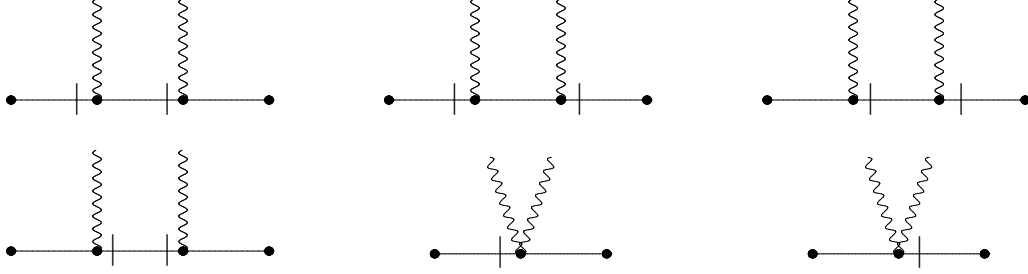
2. Subdiagrams with two chiral ends:



These diagrams are given by

$$\theta^4 \frac{M^3 \bar{D}^2}{2(\partial^2 + M^2)^3} = \theta^4 \frac{2M^2}{(\partial^2 + M^2)^2} \cdot \frac{M \bar{D}^2}{4(\partial^2 + M^2)}. \quad (171)$$

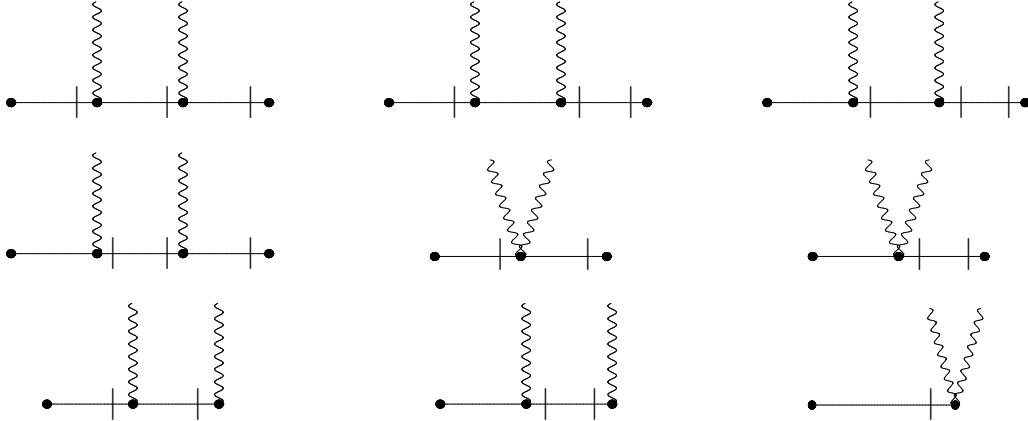
3. Subdiagrams with two antichiral ends:



These diagrams are given by

$$\theta^4 \frac{D^2 M (-\partial^2)}{2(\partial^2 + M^2)^3} = \theta^4 \left(\frac{2M^2}{(\partial^2 + M^2)^2} \cdot \frac{MD^2}{4(\partial^2 + M^2)} - \frac{MD^2}{2(\partial^2 + M^2)^2} \right). \quad (172)$$

4. Subdiagrams with an antichiral left end and a chiral right end:



These diagrams are given by

$$\theta^4 \frac{2M^2 D^2 \bar{D}^2}{(\partial^2 + M^2)^3} = \theta^4 \frac{2M^2}{(\partial^2 + M^2)} \cdot \frac{D^2 \bar{D}^2}{16(\partial^2 + M^2)}. \quad (173)$$

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